PT symmetry and the taming of instabilities

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Non-Hermitian photonics in complex media: PT symmetry and beyond

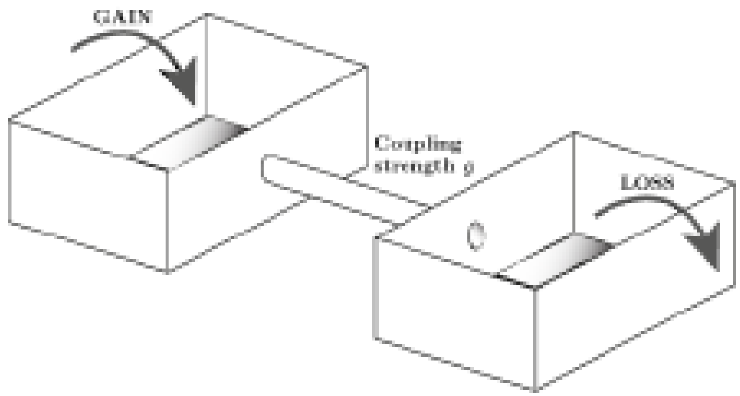
Heraklion, June 2016

PT reflection

P = space reflection
(parity)

T = time reversal

PT-symmetric systems have balanced loss and gain



If the system is in **equilibrium**, ***PT*** symmetry is **unbroken;**

If the system is **not in equilibrium**, ***PT*** symmetry is **broken**.

THE POINT OF THIS TALK:

PT-symmetric quantum theory is an extension of conventional quantum theory into the complex domain

Mathematicians have found it enlightening to extend the real number system to the complex number system

By extending conventional physical theories into the complex domain we can understand and control instabilities *and* perhaps design experiments to see this

BASIC IDEA:

If you extend real numbers to complex numbers, you lose the *ordering* property of real numbers

You give up the symbols > and <

Physical systems that look unstable might not be!

Outline

(1) Beginning

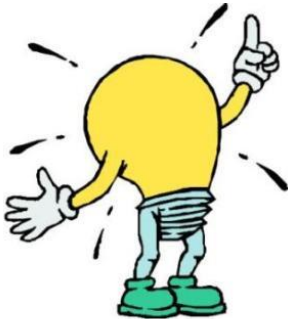
(2) Middle

(3) End

This Hamiltonian has

PT **symmetry!**

$$\boxed{H = p^2 + ix^3}$$



P = parity

T = time reversal

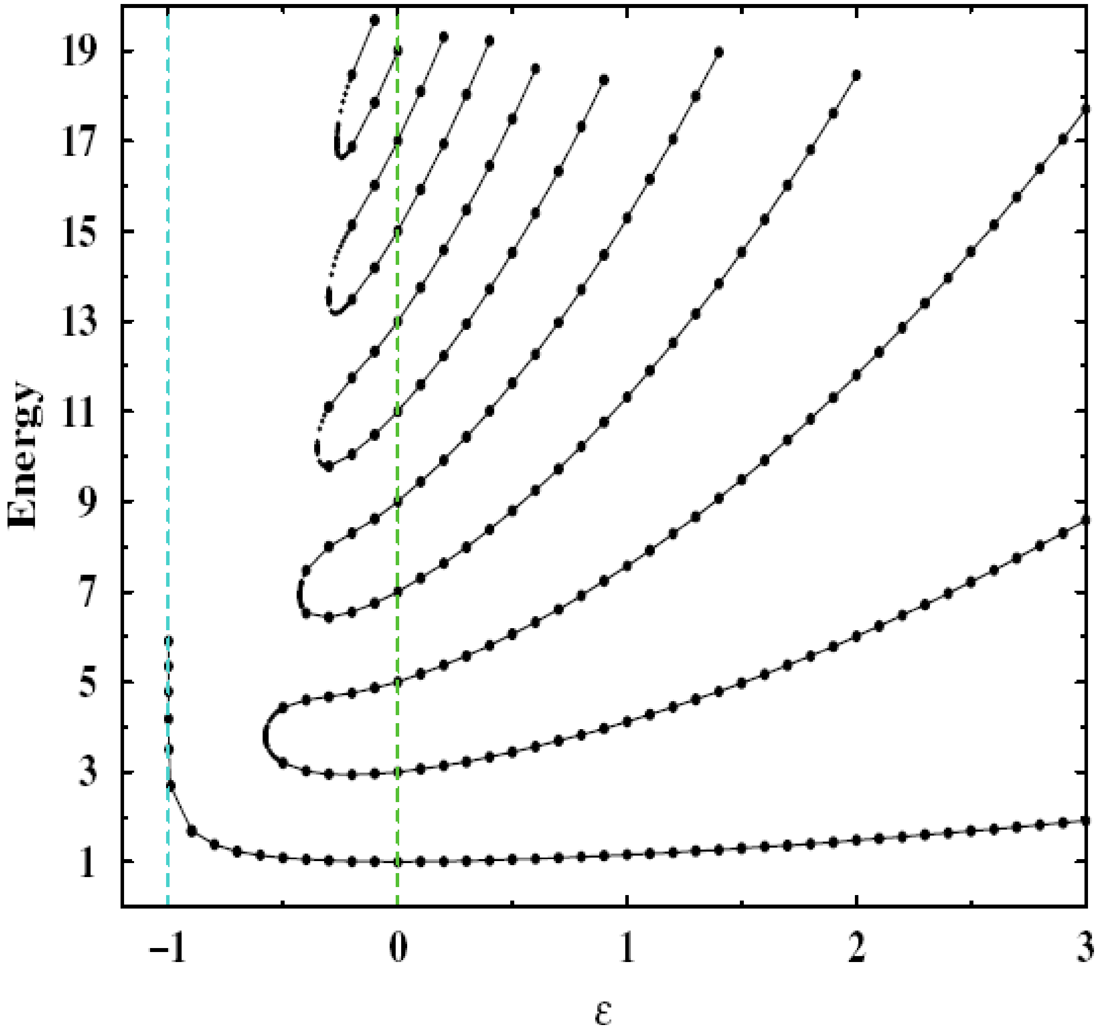
P: $x \rightarrow -x,\ p \rightarrow -p$

T: $x \rightarrow x,\ p \rightarrow -p,\ \ i \rightarrow -i$

T changes the sign of i

Big class of ***PT***-symmetric Hamiltonians:

$$\boxed{H = p^2 + x^2(ix)^\varepsilon \qquad (\varepsilon \text{ real})}$$



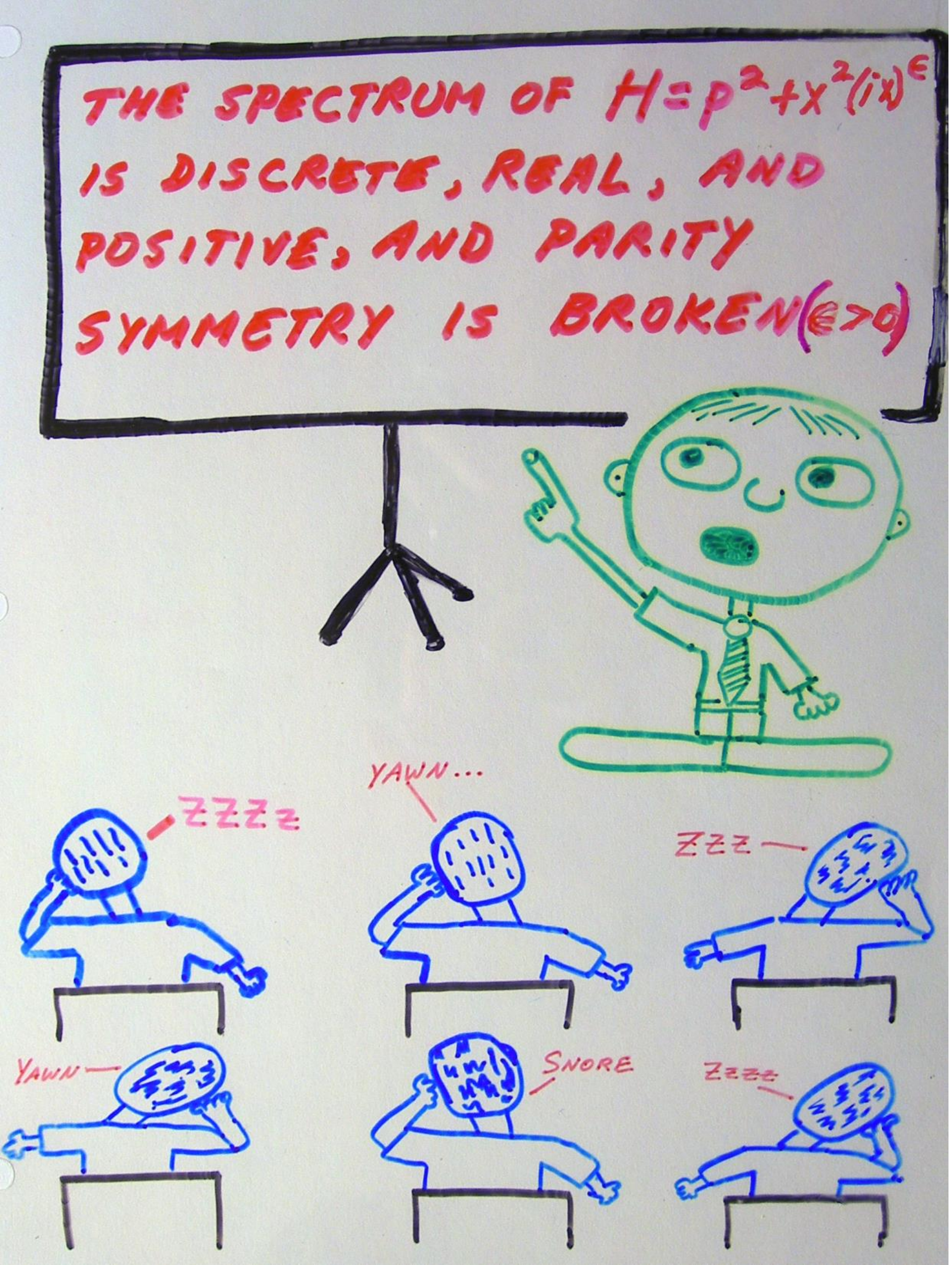
"Real spectra in non-Hermitian Hamiltonians having **PT** symmetry"
CMB and S. Boettcher
Physical Review Letters **80**, 5243 (1998)

Developments in *PT* Quantum Theory

(Since its ‘official’ beginning in 1998)

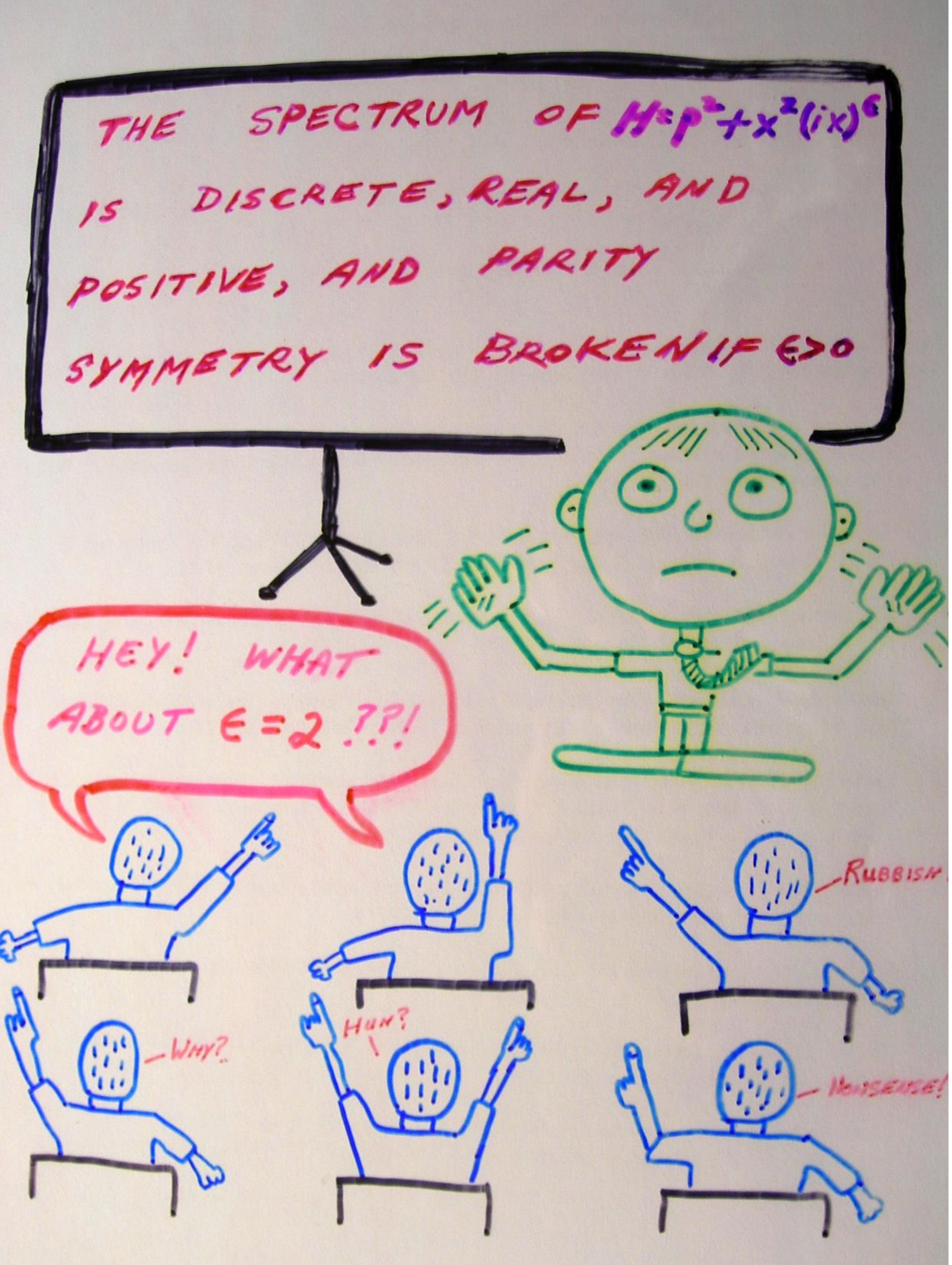
- Nearly **30** international conferences and symposia
- 2000 published papers
- Many experimental results in past seven or eight years

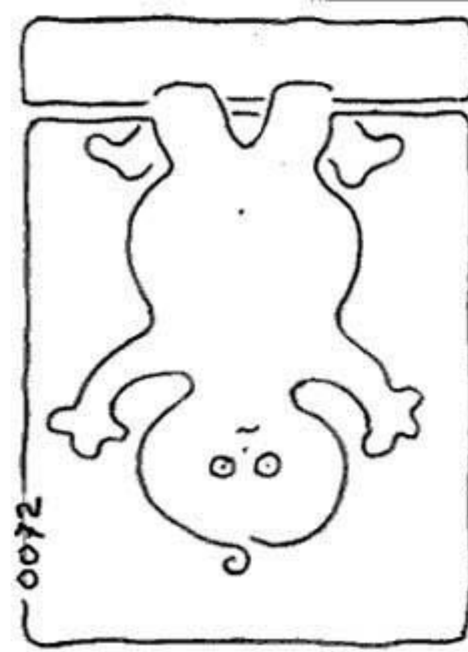
***PT* is doing well!**



Rigorous proof of real eigenvalues:

"ODE/IM correspondence,"
P. Dorey, C. Dunning, and R. Tateo,
J. Phys. A **40**, R205 (2007)





Unstable upside-down potential
With real positive eigenvalues?!

$$V(x) = -x^4$$

Z. Ahmed, CMB, and M. V. Berry,
J. Phys. A: Math. Gen. **38**, L627 (2005)

CMB, D. C. Brody, J.-H. Chen, H. F. Jones,
K. A. Milton, and M. C. Ogilvie,
Phys. Rev. D **74**, 025016 (2006)

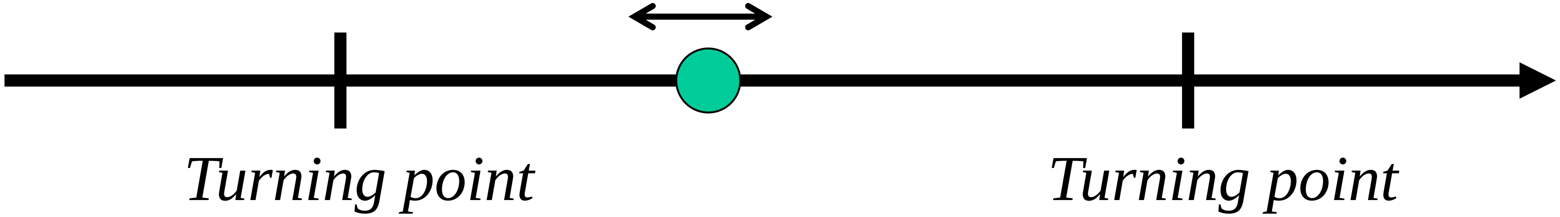
$$V(x) = -x^4$$

This potential is unstable! (*<u>on the real axis</u>*)

How can it possibly have bound states?

Classical harmonic oscillator

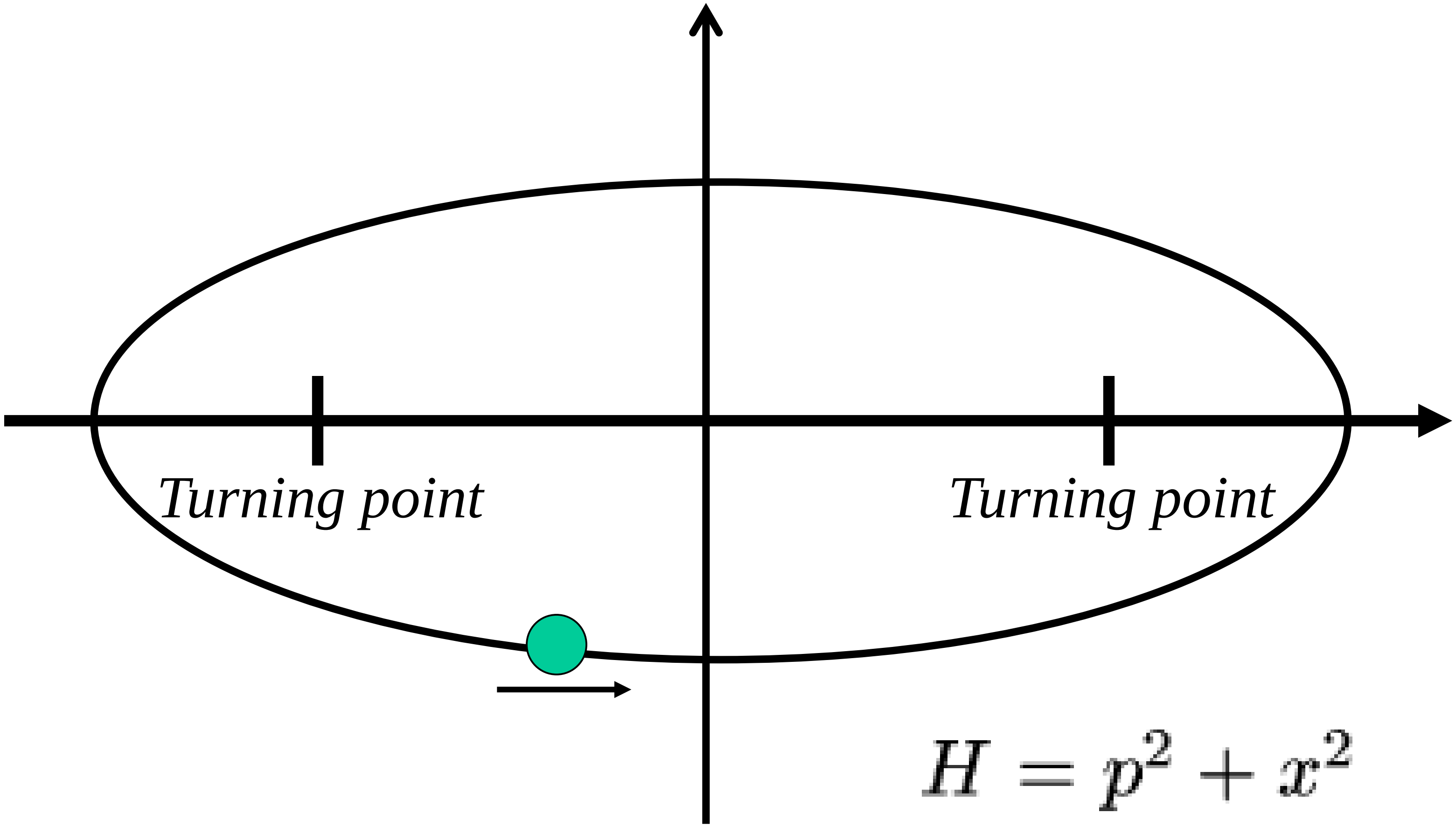
Back and forth motion on the real axis:



$$H = p^2 + x^2 \qquad (\epsilon = 0)$$

Classically allowed and classically forbidden regions

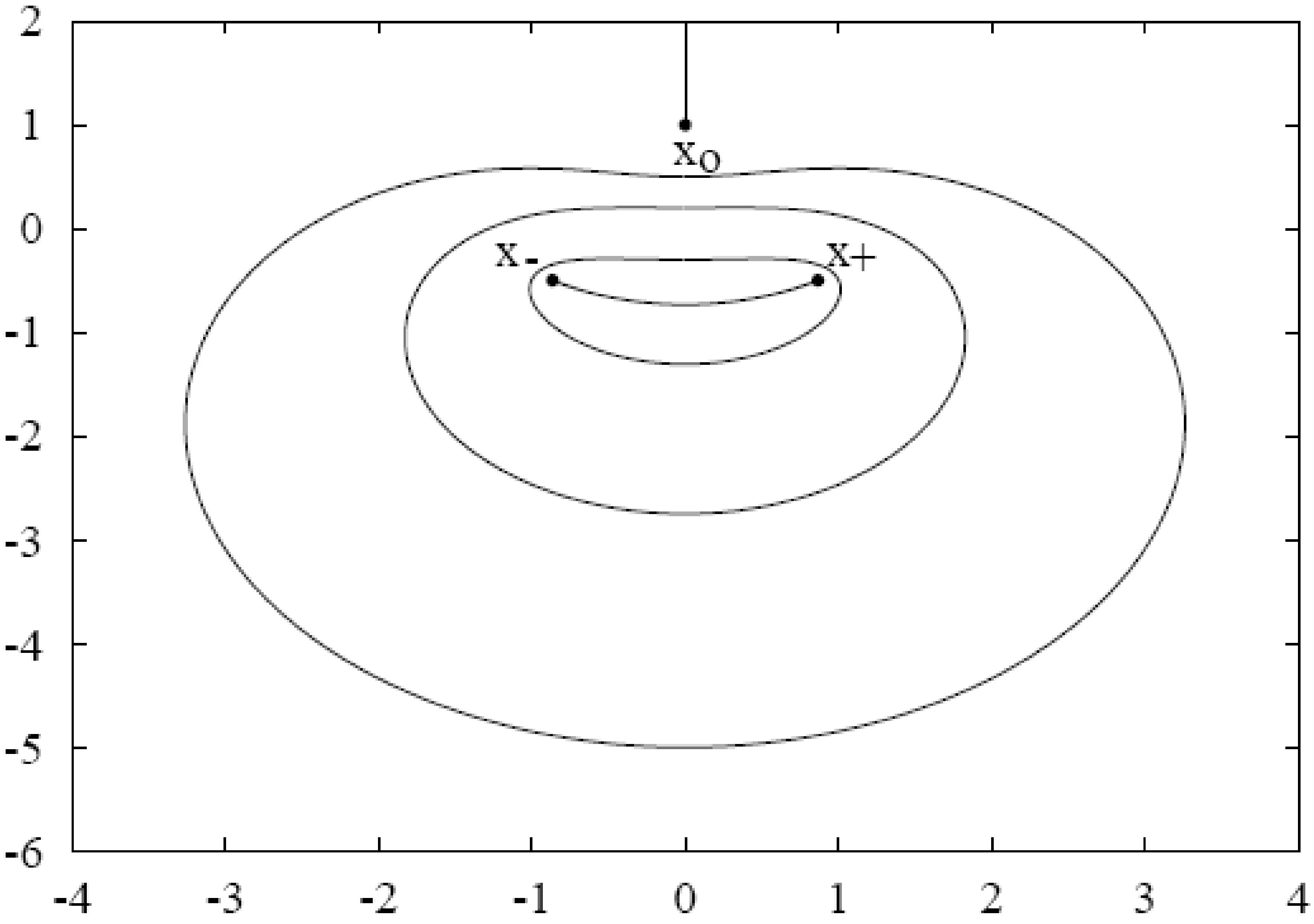
Harmonic oscillator in complex plane



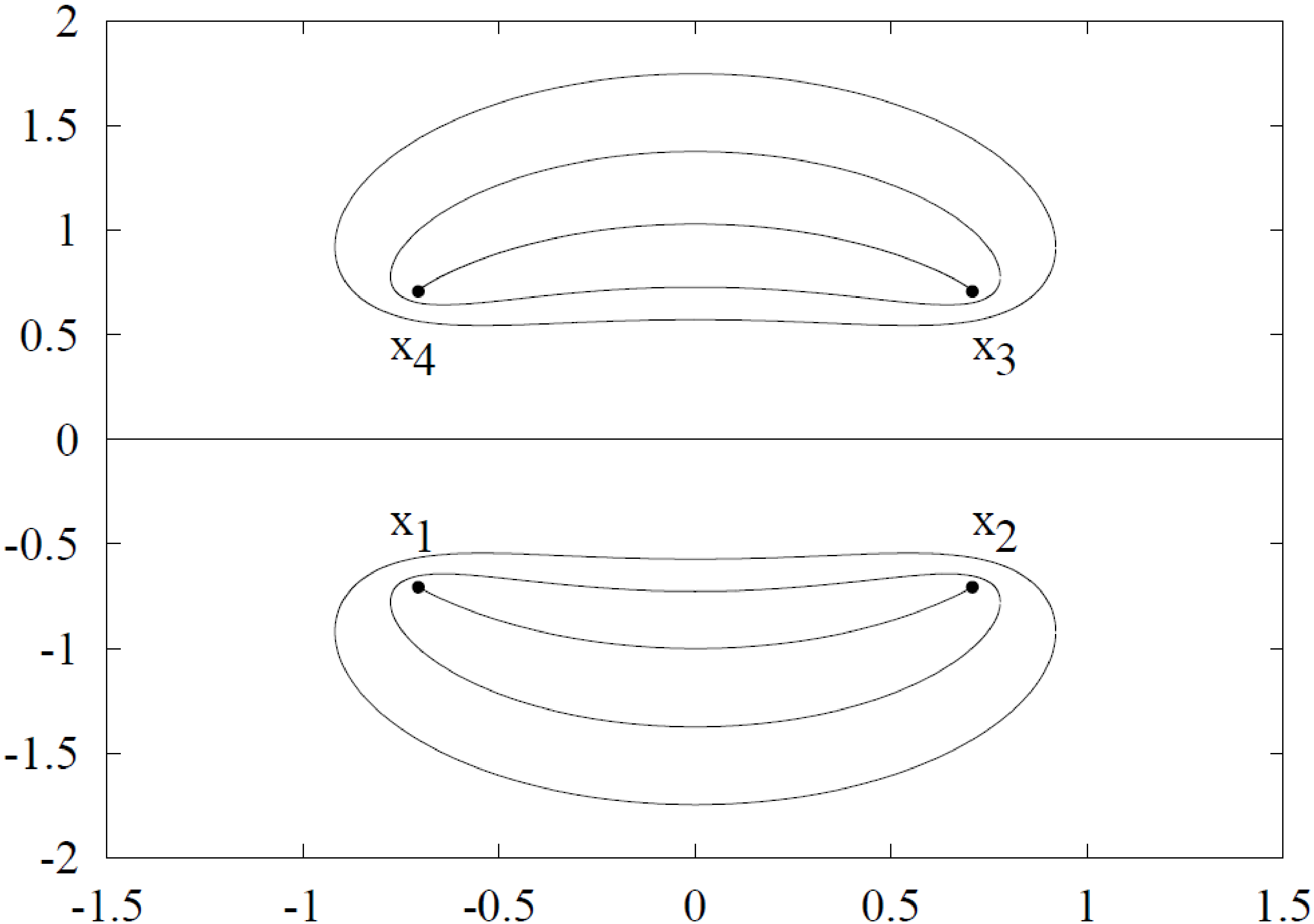
$$H = p^2 + x^2$$

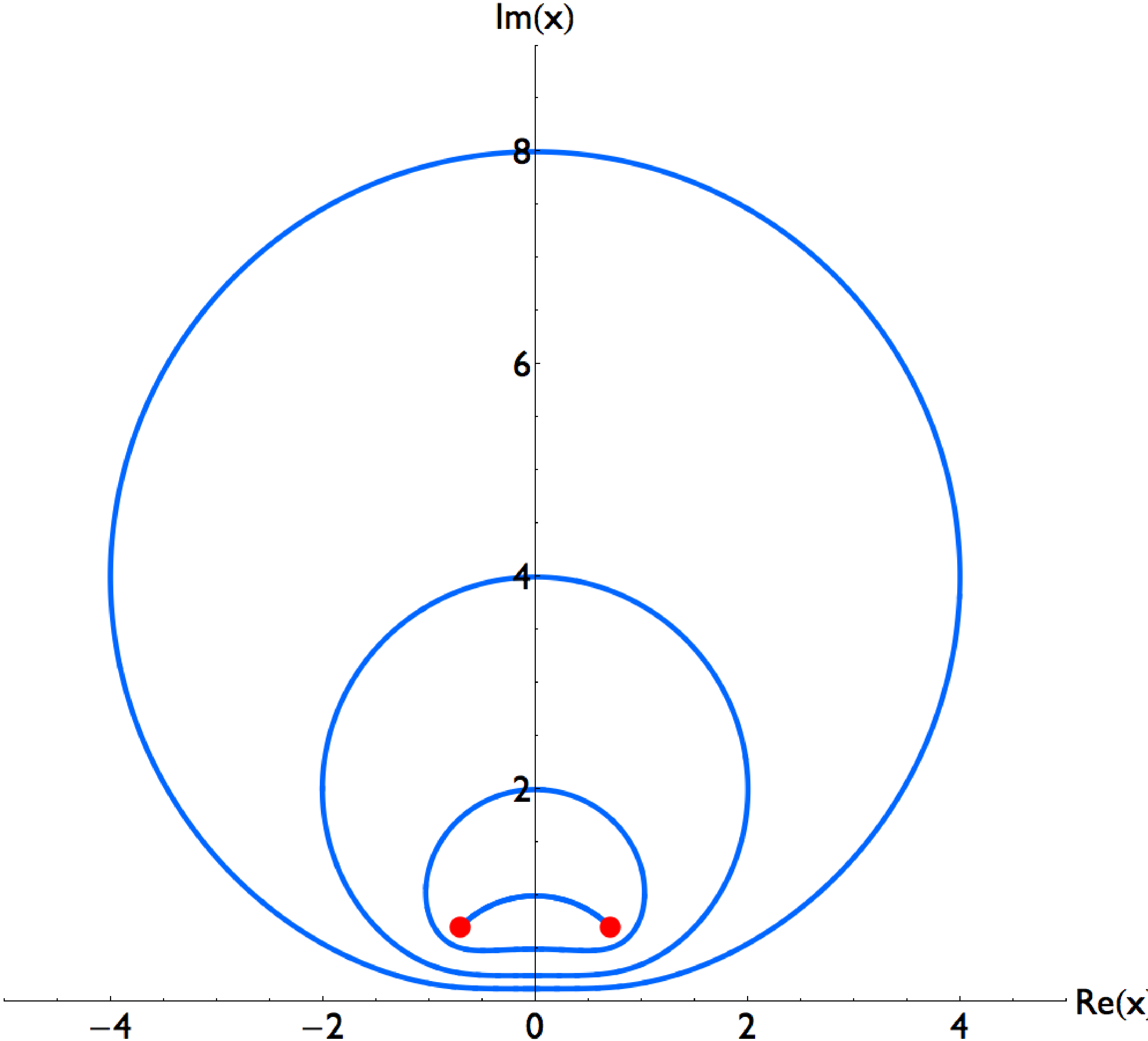
($\epsilon = 0$)

$$H = p^2 + ix^3 \qquad (\varepsilon = 1)$$

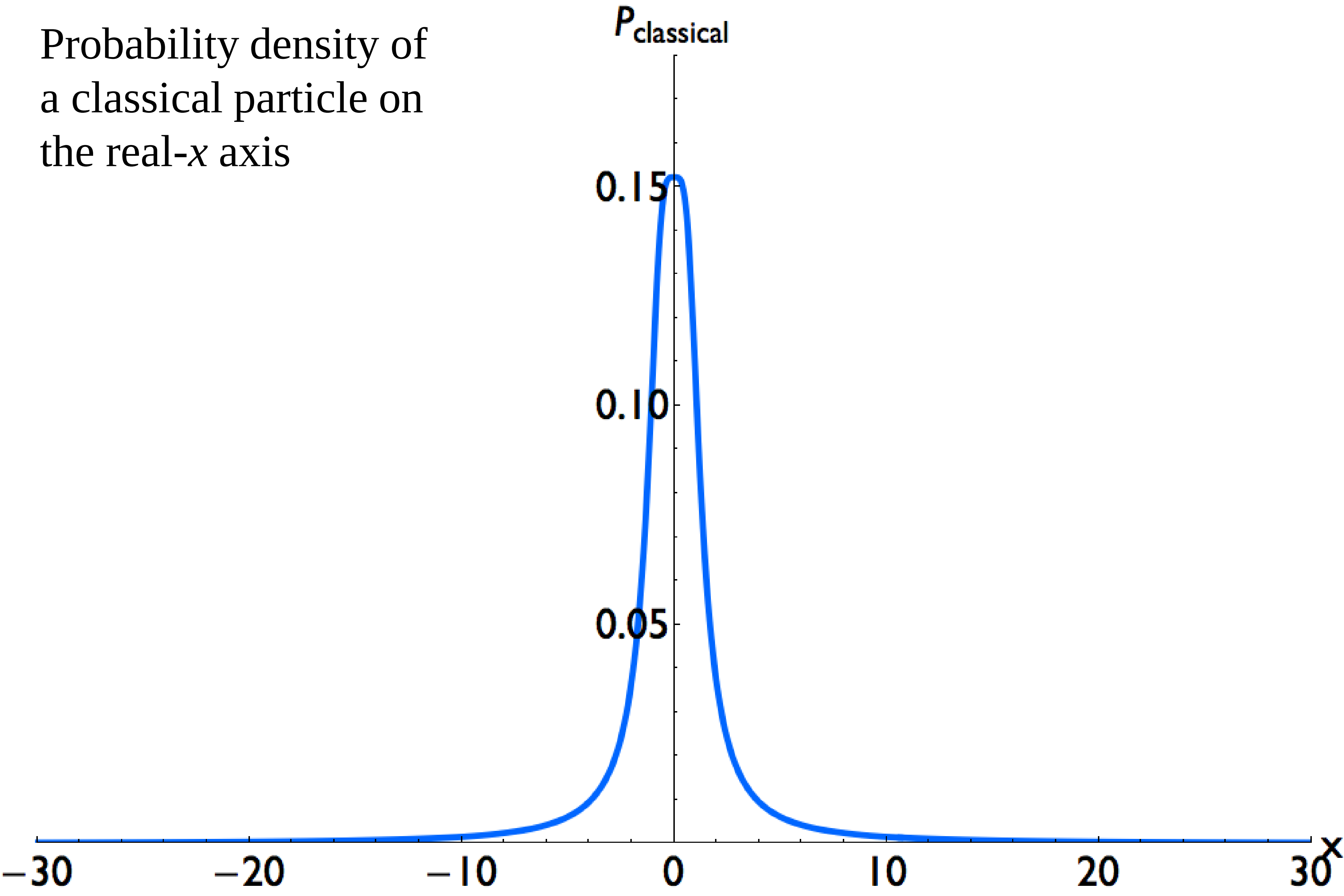


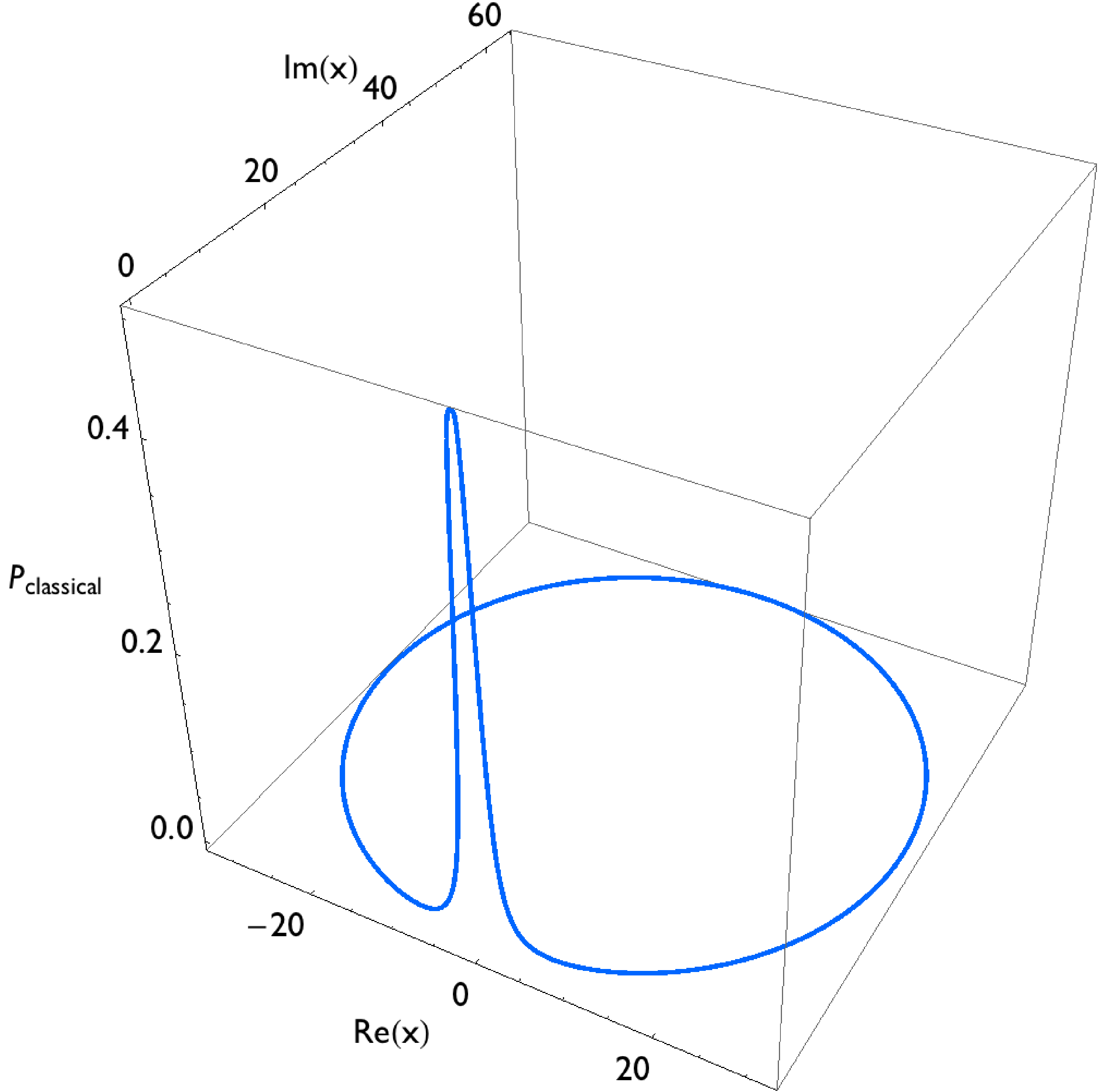
$$H = p^2 - x^4 \qquad (\varepsilon = 2)$$




Im(x)
8
6
4
2
Re(x)
−4
−2
0
2
4

Probability density of a classical particle on the real-*x* axis



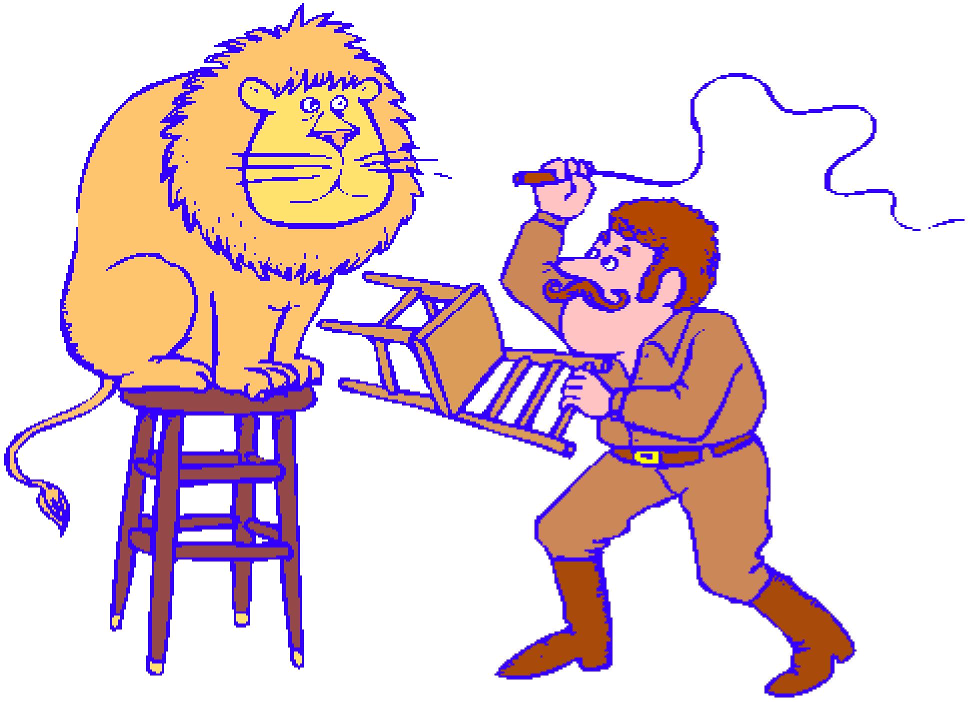


Im(x)
60
40
20
0
0.4
$P_{\text{classical}}$
0.2
0.0
−20
0
Re(x)
20

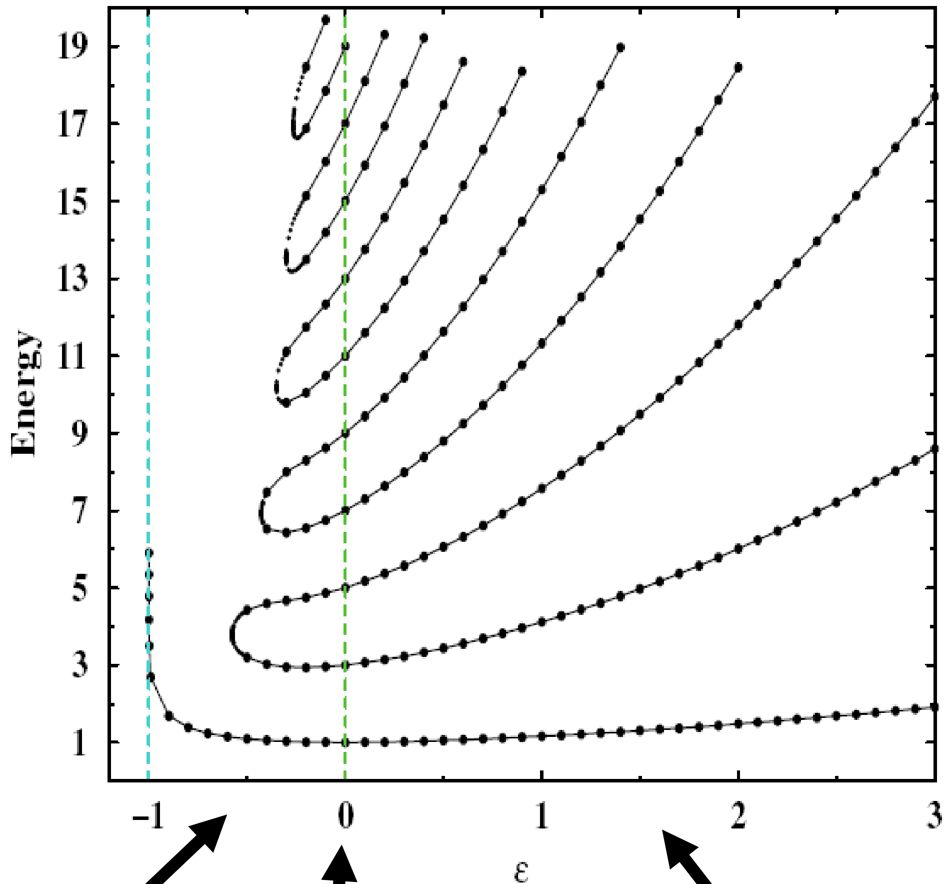
Bohr-Sommerfeld Quantization of a complex atom

$$\oint dx\, p = \left(n + \frac{1}{2}\right)\pi$$

The instability at $x = 0$ is tamed!



$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real})$$



Transition

at ε = 0

Region of ***broken***
PT symmetry

PT **Boundary**

Region of ***unbroken***
PT symmetry

Broken ***P***arro***T*** Unbroken ***P***arro***T***

The condition of *PT* symmetry is weaker than the condition of Hermiticity. All eigenvalues *E* of a Hermitian Hamiltonian are real. For *PT*-symmetric Hamiltonians *only the secular equation* det(H - IE) = 0 *is real*. Thus,

Unlike Hermitian Hamiltonians, there are *TWO POSSIBILITIES:*

PT-symmetric theories may have an *all* real or a *partially* real spectrum

Two possibilities

Hermitian Hamiltonians: BORING!

Eigenvalues are always real – nothing interesting happens



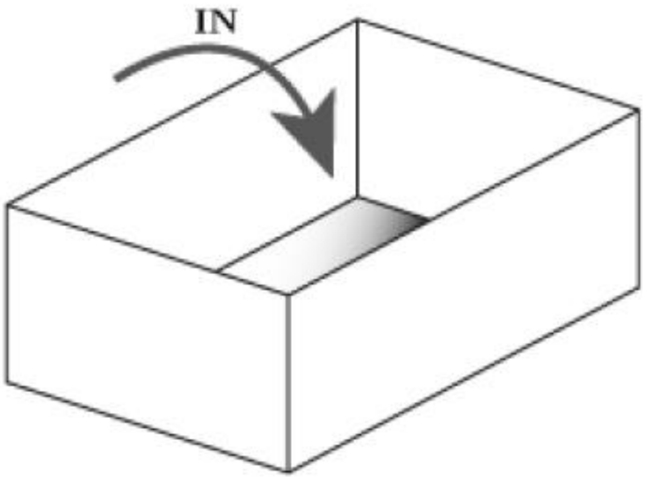
PT-symmetric Hamiltonians: ASTONISHING!

Transition between parametric regions of
broken and unbroken *PT* symmetry.
Can be observed experimentally!

Intuitive explanation of *PT* transition …

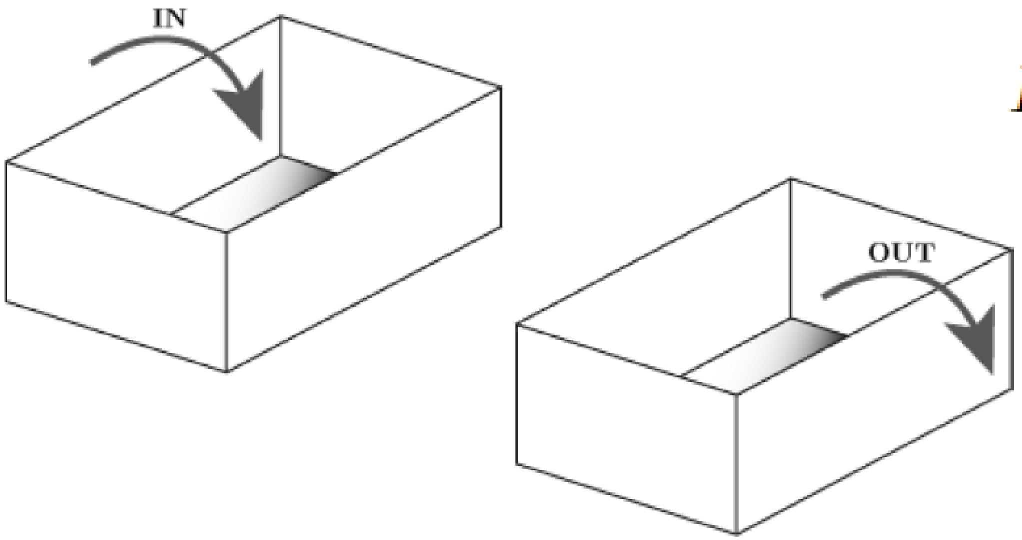
Imagine a closed box with gain… Hamiltonian for this system is non-Hermitian:

$$H = [a + ib]$$



Box 1: Gain

Two noninteracting closed boxes, one with gain and the other with loss:



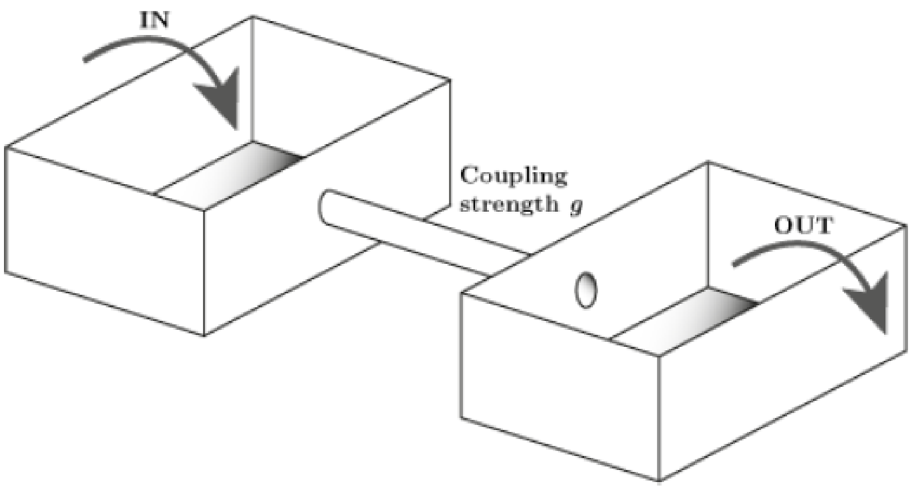
$$H_{\text{combined}} = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$$

Box 1: Gain

Box 2: Loss

Hamiltonian for this system is non-Hermitian but *PT* symmetric, but the system is not in equilibrium

Now, couple the boxes:



$$H_{\text{coupled}} = \begin{bmatrix} a+ib & g \\ g & a-ib \end{bmatrix}$$

Box 1: Gain

Box 2: Loss

This system is in equilibrium for sufficiently large coupling

2 x 2 Non-Hermitian matrix *PT*-symmetric Hamiltonian

Real secular equation:

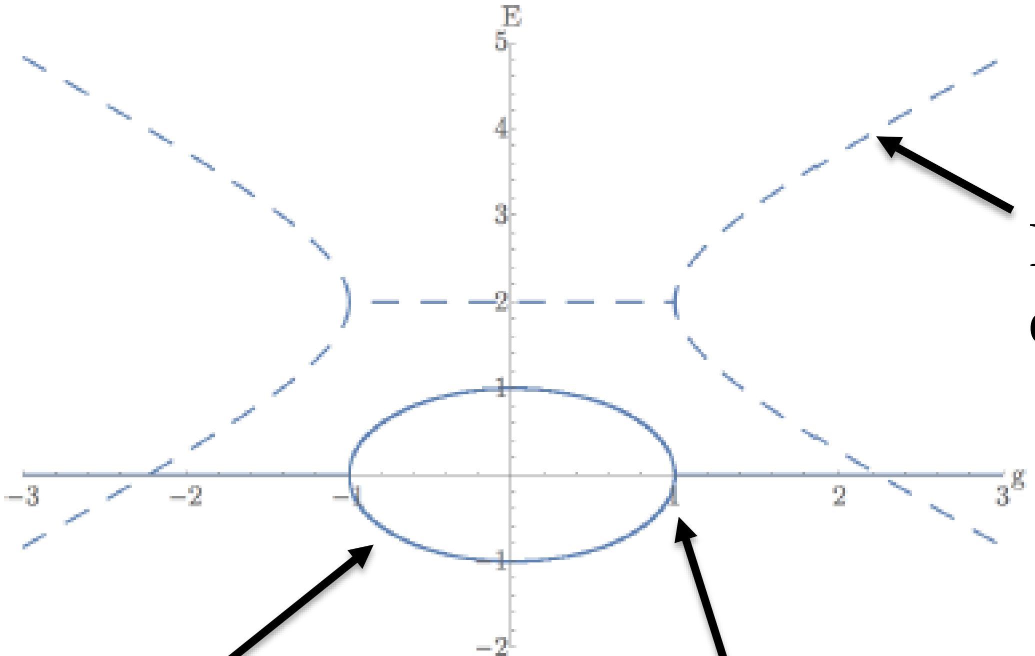
$$\det\left(H_{\text{coupled}} - IE\right) = E^2 - 2aE + a^2 + b^2 - g^2$$

$$E_{\pm} = a \pm \sqrt{g^2 - b^2}$$

Time reversal: $\mathcal{T}$ = complex conjugation

Parity: $\mathcal{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$a = 2$, $b = 1$



Real part
of the energy

Imaginary part
of the energy

Transition
at $|g| = 1$

PT-symmetric systems lie between closed and open systems

Hermitian H

PT-symmetric H

Non-Hermitian H

***PT* symmetry *controls instabilities*!**

Physical systems that you might think are unstable become *stable* in the complex domain!

Electromagnetic self-force and runaway modes

$$\tau \dddot{x} - \ddot{x} - kx = 0,$$
$$-\tau \dddot{y} - \ddot{y} - ky = 0.$$

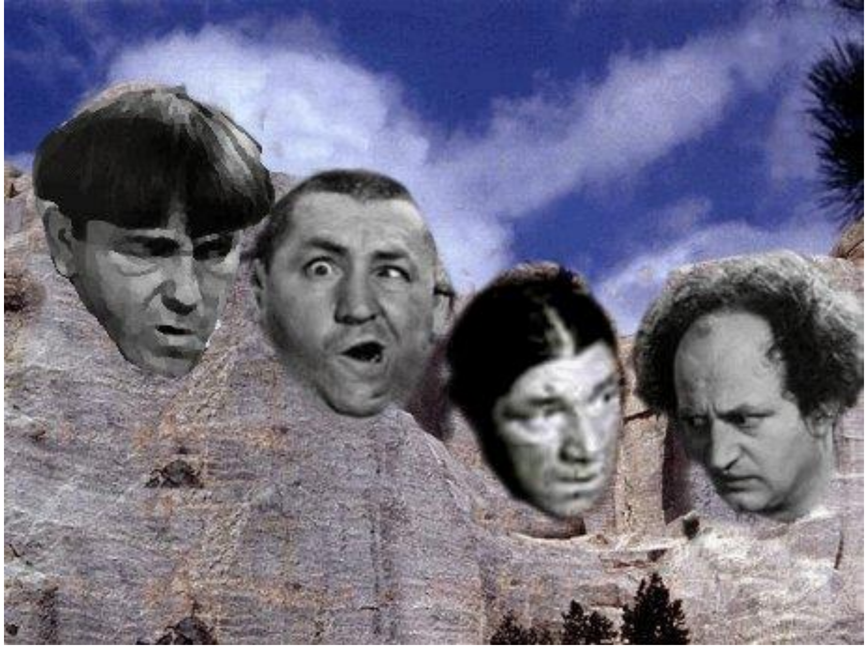
$$H = \frac{1}{\tau}(ps - rq) + \frac{1}{\tau^2} rs + \frac{1}{2}(pz + qw) - \frac{1}{4} zw + kxy.$$

	x	y	z	w	p	q	r	s
$\mathcal{P}$	y	x	w	z	q	p	s	r
$\mathcal{T}$	x	y	-z	-w	-p	-q	r	s
$\mathcal{PT}$	y	x	-w	-z	-q	-p	s	r

“**PT**-symmetric interpretation of the electromagnetic self-force,” CMB and M. Gianfreda, J. Phys. A: Math. Theor. **48**, 34FT01 (2015)

Taming an instability

Four examples of instability problems



1. Lee model

2. Pais-Uhlenbeck oscillator

3. Double-scaling limit in QFT

4. Nonlinear differential-equation eigenvalue problems

Example 1: Lee model

$$V \to N + \theta, \qquad N + \theta \to V.$$

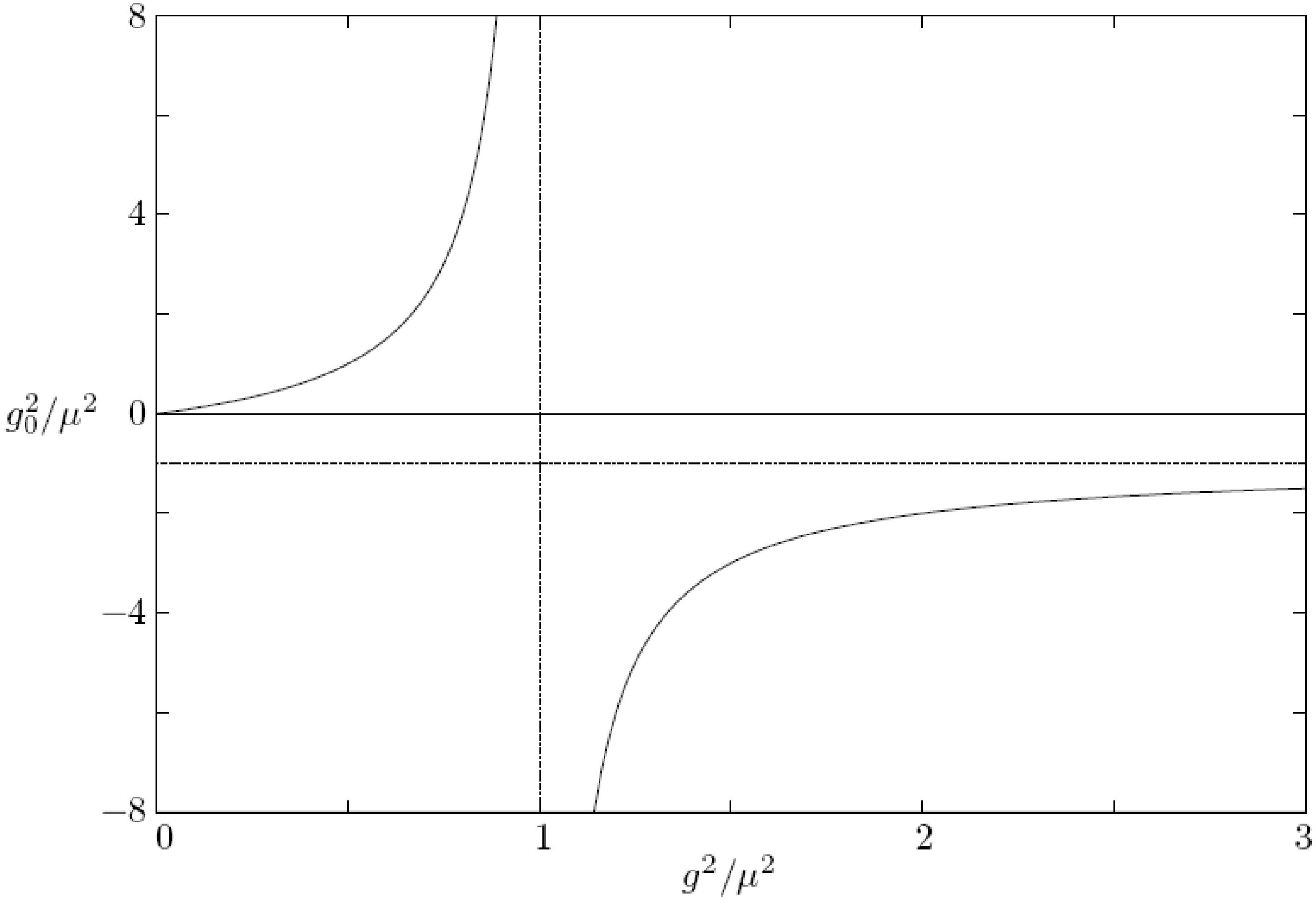
$$H = H_0 + g_0 H_1,$$

$$\begin{aligned} H_0 &= m_{V_0} V^\dagger V + m_N N^\dagger N + m_\theta a^\dagger a, \\ H_1 &= V^\dagger N a + a^\dagger N^\dagger V. \end{aligned}$$

T. D. Lee, Phys. Rev. **95**, 1329 (1954)

G. Källén and W. Pauli, Dan. Mat. Fys. Medd. **30**, No. 7 (1955)

Problem with the Lee model



$$g_0^2 = g^2 / \left(1 - g^2/\mu^2\right)$$

MR0076639 (17,927d) 81.0X

Källén, G.; **Pauli, W.**

On the mathematical structure of T. D. Lee's model of a renormalizable field theory.

Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), *no. 7,* 23 *pp.*

Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR0064658 (16,317b)] has recently suggested perhaps the first non-trivial model of a field-theory which can be explicitly solved. Three particles (V, N and θ) are coupled, the explicit solution being secured by allowing reactions $V \rightleftarrows N+\theta$ but forbidding $N \rightleftarrows V+\theta$. The theory needs conventional mass and charge renormalizations which likewise can be explicitly calculated. The renormalized coupling constant g is connected to the unrenormalized constant g_0 by the relation $g^2/{g_0}^2 = 1 - Ag^2$, where A is a divergent integral. This can be made finite by a introducing a cut-off.

The importance of Lee's result lies in the fact that Schwinger (unpublished) had already proved on very general principles, that the ratio $g^2/{g_0}^2$ should lie between zero and one. [For published proofs of Schwinger's result, see Umezawa and Kamefuchi, Progr. Theoret. Phys. **6** (1951), 543–558; MR0046306 (13,713d); Källén, Helv. Phys. Acta **25** (1952), 417–434; MR0051156 (14,435l); Lehmann, Nuovo Cimento (9) **11** (1954), 342–357; MR0072756 (17,332e); Gell-Mann and Low, Phys. Rev. (2) **95** (1954), 1300–1312; MR0064652 (16,315e)]. The results of Lee and Schwinger can be reconciled only if (i) there is a cut-off in Lee's theory and (ii) if g lies below a critical value g_{crit}. The present paper is devoted to investigation of physical consequences if these two conditions are not satisfied.

The authors discover the remarkable result that if $g > g_{\text{crit}}$ there is exactly one new eigenstate for the physical V-particle having an energy that is below the mass of the normal V-particle. It is further shown that the S-matrix for Lee's theory is not unitary when $g > g_{\text{crit}}$ and that the probability for an incoming V-particle in the normal state and a θ-meson, to make a transition to an outgoing V-particle in the new ("ghost") state, must be negative if the sum of all transition probabilities for the in-coming state shall add up to one. The possible implication of Källén and Pauli's results for quantum-electrodynamics, where in perturbation theory $(e/e_0)^2$ has a behaviour similar to $(g/g_0)^2$ in Lee's theory, need not be stressed.

Reviewed by *A. Salam*

“A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation.”

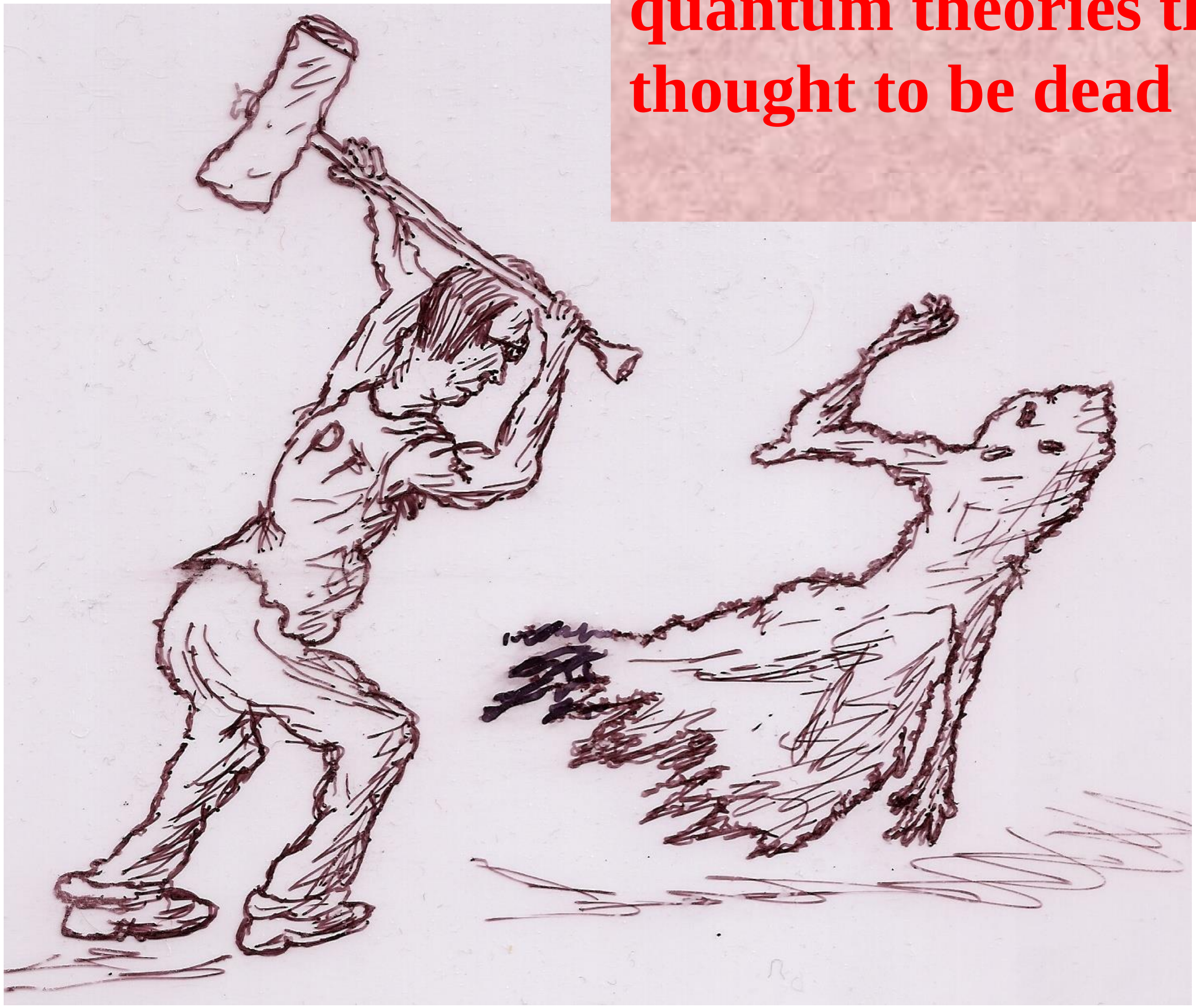
G. Barton, *Introduction to Advanced Field Theory* (John Wiley & Sons, New York, 1963)

This is a *really* hard problem. Pauli, Heisenberg, Wick, Sudarshan, … worked on it, but no cigar

A really hard problem

The solution…

GHOSTBUSTING: Reviving quantum theories that were thought to be dead



"Ghost busting: ***PT***-symmetric interpretation of the Lee model,"
CMB, S. Brandt, J.-H. Chen, and Q. Wang, *Phys. Rev. D* **71**, 025014 (2005)

Example 2: Pais-Uhlenbeck model

$$I = \frac{\gamma}{2} \int dt \left[\ddot{z}^2 - \left(\omega_1^2 + \omega_2^2 \right) \dot{z}^2 + \omega_1^2 \omega_2^2 z^2 \right]$$

Gives a fourth-order field equation:

$$z''''(t) + (\omega_1^2 + \omega_2^2) z''(t) + \omega_1^2 \omega_2^2 z(t) = 0$$

Hard problem: A fourth-order field equation gives a propagator like

$$G(E)=\frac{1}{(E^2+m_1^2)(E^2+m_2^2)}$$

$$G(E)=\frac{1}{m_2^2-m_1^2}\left(\frac{1}{E^2+m_1^2}-\frac{1}{E^2+m_2^2}\right)$$

GHOST!

We looked VERY hard for a solution…
And found it!

Solution: This is a *PT*-symmetric theory and there are no ghosts!

CMB and P. D. Mannheim,
"No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model,"
Physical Review Letters 100, 110402 (2008)

Example 3:
Double-scaling limit in QFT

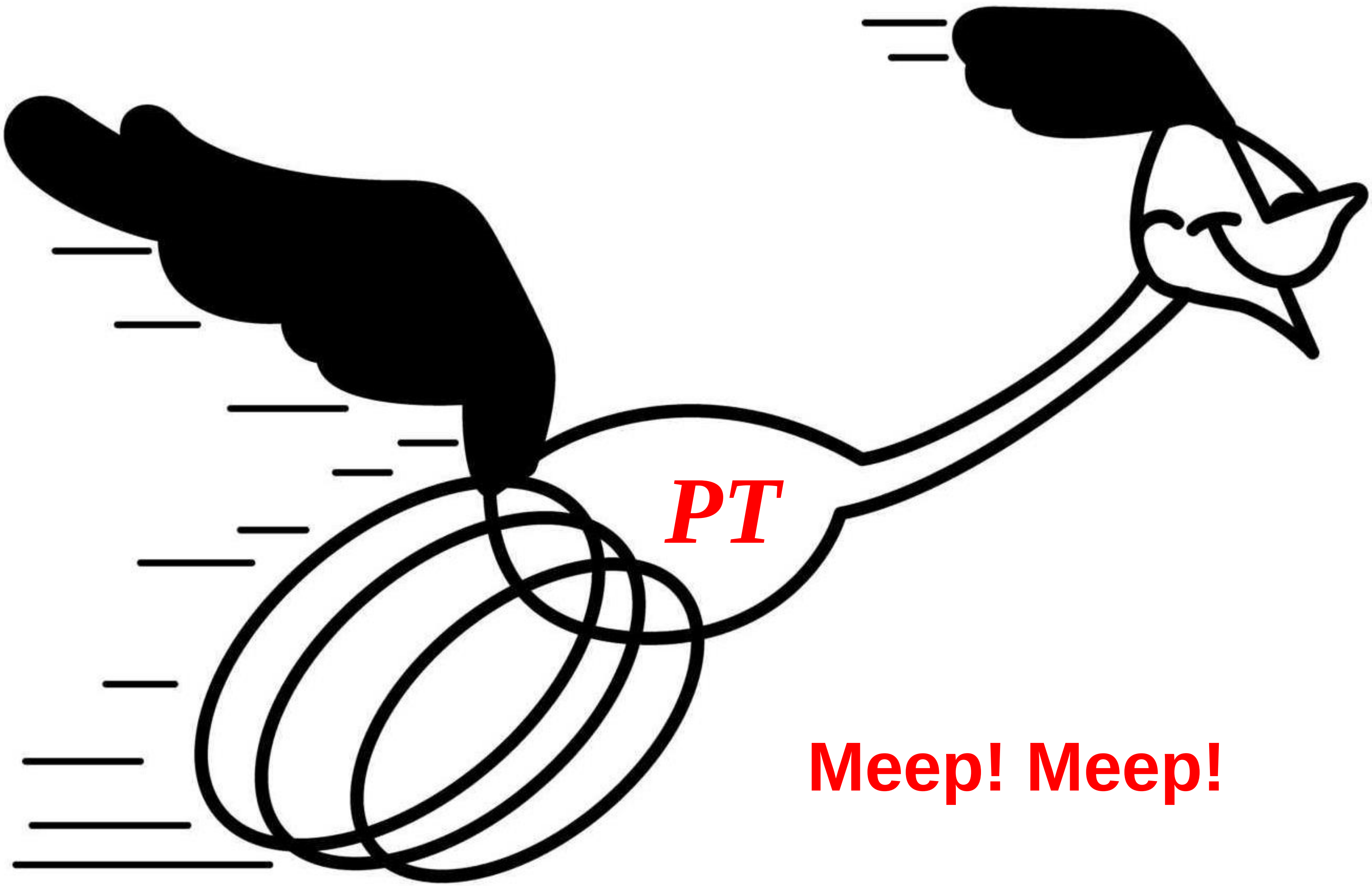
The double-scaling limit is a *correlated limit*; it is universal and produces an entire function of scaled variable

BUT!
The double-scaling limit of a conventional quartic theory gives a “wrong-sign” (*upside-down* potential) theory: $-g\phi^4$

CMB, M. Moshe, and S. Sarkar, *J. Phys. A: Math. Theor.* **46**, 102002 (2013)

CMB and S. Sarkar, J. Phys. A: Math. Theor. **46**, 442001 (2013)

PT-symmetric quantum mechanics to the rescue!



Meep! Meep!

Example 4: Instabilities of nonlinear differential equations

Painlevé transcendents have fundamental instabilities that can be tamed and understood quantitatively by using *PT*-symmetric quantum theory

"Nonlinear eigenvalue problems,"
CMB, A. Fring, and J. Komijani,
Journal of Physics A: Mathematical and Theoretical **47**, 235204 (2014)

"Painleve Transcendents and **PT**-Symmetric Hamiltonians"
CMB and J. Komijani
Journal of Physics A: Mathematical and Theoretical **48**, 475202 (2015)

Instability of Painlevé I explained from large eigenvalues of

cubic PT-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 + 2ix^3$$

Painlevé I corresponds to ε = 1

(Do you remember the cubic ***PT***-symmetric Hamiltonian?!)

Instability of Painlevé II explained from large eigenvalues of

quartic PT-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 - \frac{1}{2}x^4$$

Painlevé II corresponds to ε = 2

(Do you remember the quartic upside-down ***PT***-symmetric Hamiltonian?!)

Instability of Painlevé IV explained in terms of the

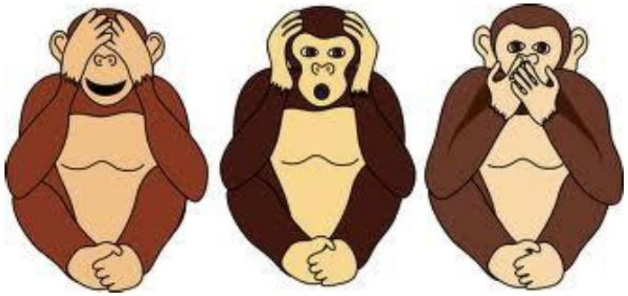
sextic PT-symmetric Hamiltonian

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{8}\hat{x}^6.$$

Painlevé IV corresponds to ε = 4

(Do you remember the sextic ***PT***-symmetric Hamiltonian?!)

Recent *PT* research problem:



Stability of the Higgs vacuum

Instability of the Higgs vacuum

Standard Model of particle physics: φ is the Higgs field.
Effective potential is $\Gamma[\varphi] \propto -\varphi^4 \log(\varphi^2)$

"Electroweak Higgs potentials and vacuum stability,"
M. Sher, *Phys. Repts.* **179**, 273-418 (1989)

Let's examine the model $H = p^2 - x^4 \log(x^2)$

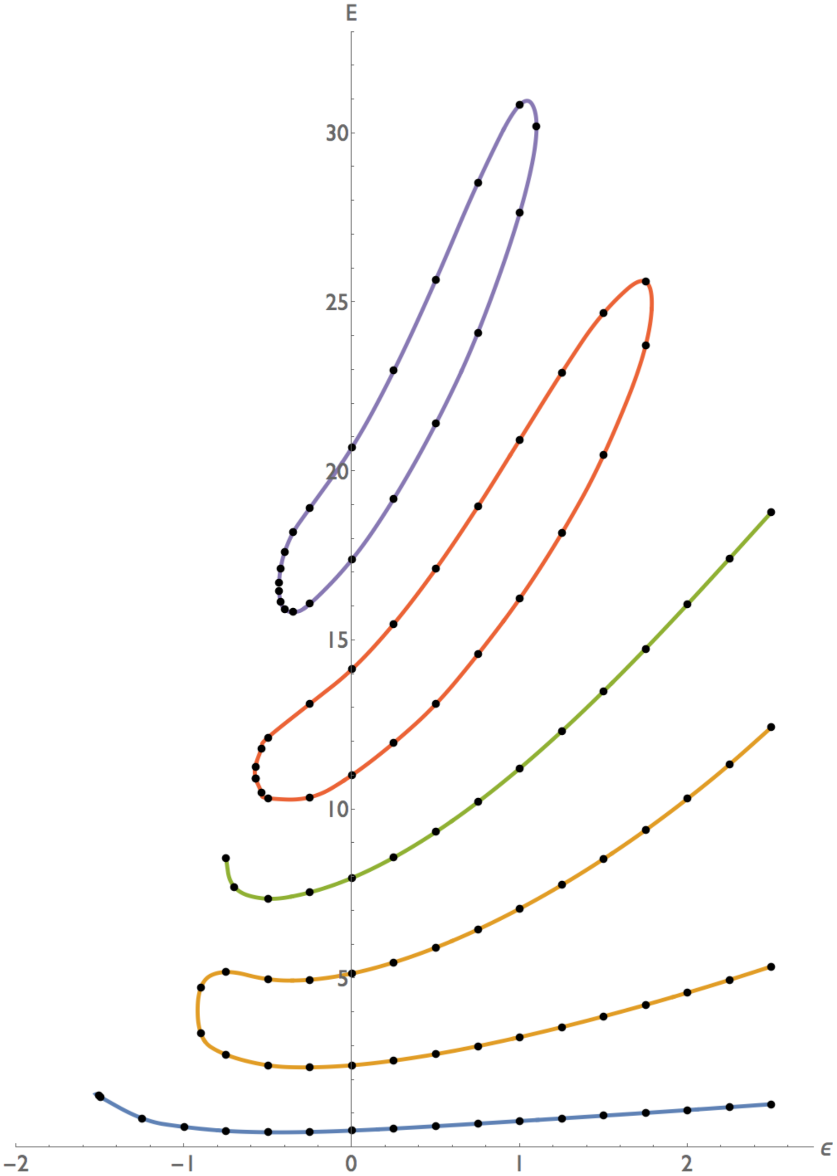
$H = p^2 + x^2(ix)^\epsilon \log\left(x^2\right)$ $\qquad\qquad \epsilon : 0 \to 2$

This is a **REALLY** tough problem…

Really tough problems

$$H = p^2 + x^2(ix)^\epsilon \log(x^2)$$

$$\epsilon \rightarrow 2$$



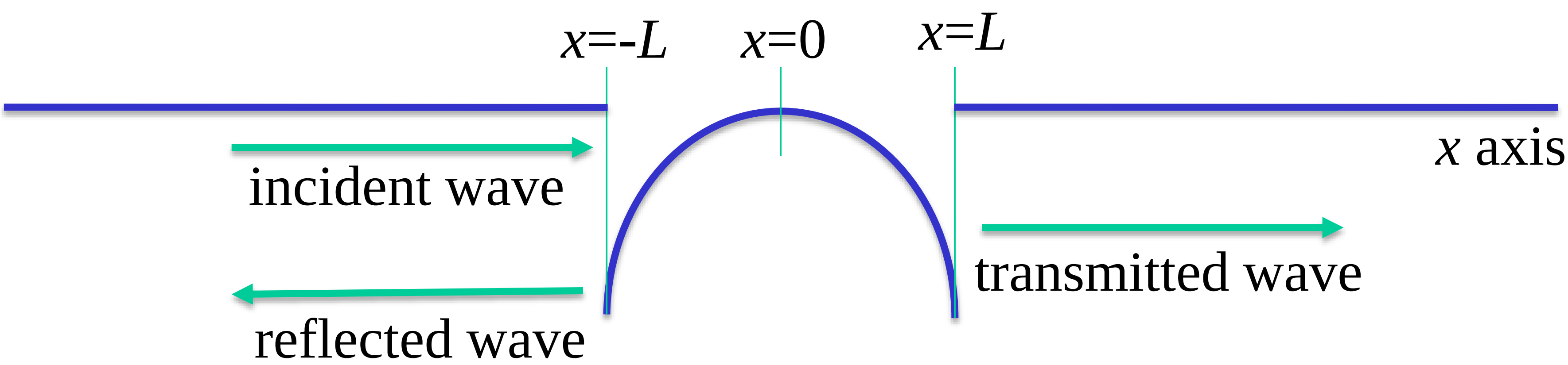
Only four real eigenvalues at ε = 2.

Maybe the world is in a broken ***PT***-symmetric state!

CMB, D. W. Hook, N. Mavromatos, and S. Sarkar, arXiv: 1506.01970

Proposal for a quantum experiment:

REGION I: $x<-L$ REGION II: $-L<x<L$ REGION III: $x>L$



$$V(x) = 0 \qquad V(x) = -x^4 \qquad V(x) = 0$$

Schrödinger equation

$$i\psi_t(x,t) = -\psi_{xx}(x,t) + V(x)\psi(x,t)$$

Let $\quad \psi(x,t) = e^{-iEt}\phi(x)$

Region I: Right-going *incident plane wave*
+ Left-going *reflected plane wave*

$$\phi_{\mathrm{I}}(x) = e^{ix\sqrt{E}} + Re^{-ix\sqrt{E}}$$

Region III: Right-going *transmitted plane wave*

$$\phi_{\mathrm{III}}(x) = Te^{ix\sqrt{E}}$$

Define: $y(x) \equiv \dfrac{\phi(x)}{Te^{iL\sqrt{E}}}$

Initial conditions at $x = L$:

$$y_{\mathrm{III}}(L) = 1$$

$$y'_{\mathrm{III}}(L) = i\sqrt{E}$$

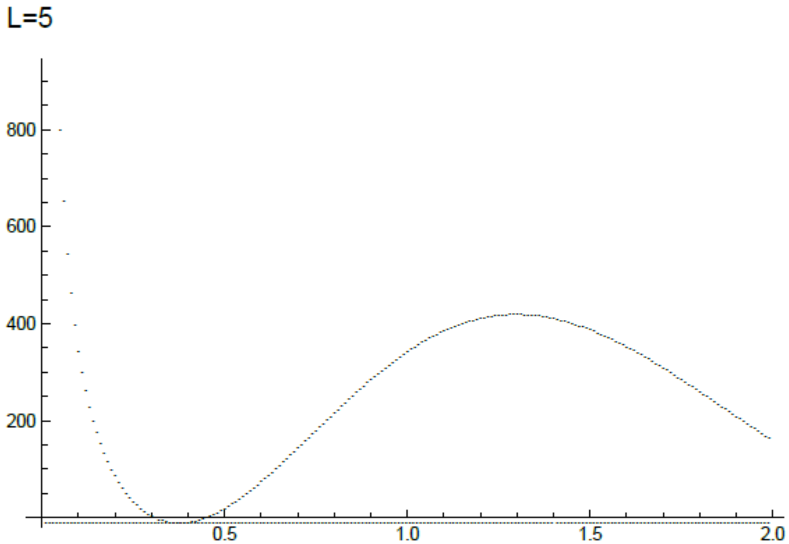
Integrate from $x = L$ down to $x = -L$

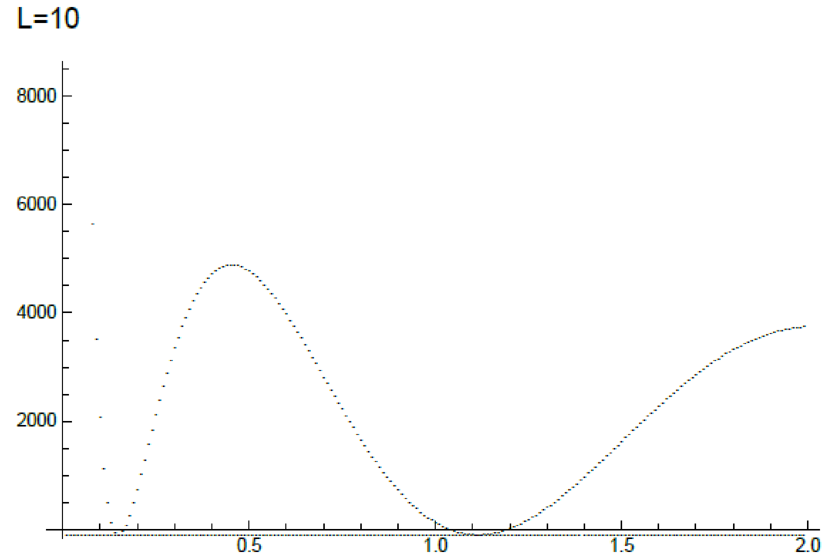
$$y_{\mathrm{I}}(-L) = \frac{1}{T} e^{-2iL\sqrt{E}} + \frac{R}{T}$$

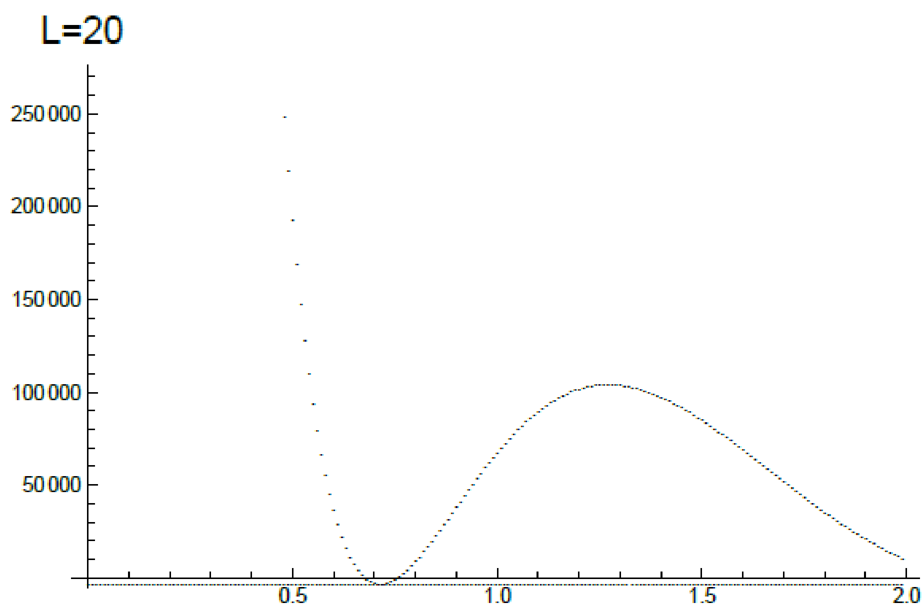
$$y_{\mathrm{I}}'(-L) = \frac{i\sqrt{E}}{T} e^{-2iL\sqrt{E}} - \frac{iR\sqrt{E}}{T}$$

$$\left| y_{\mathrm{I}}(-L) - \frac{y_{\mathrm{I}}'(-L)}{i\sqrt{E}} \right| = 2\left|\frac{R}{T}\right|$$

Incident plane wave – plot $|R/T|$







It doesn't work!Depends on *L*

Why??!!

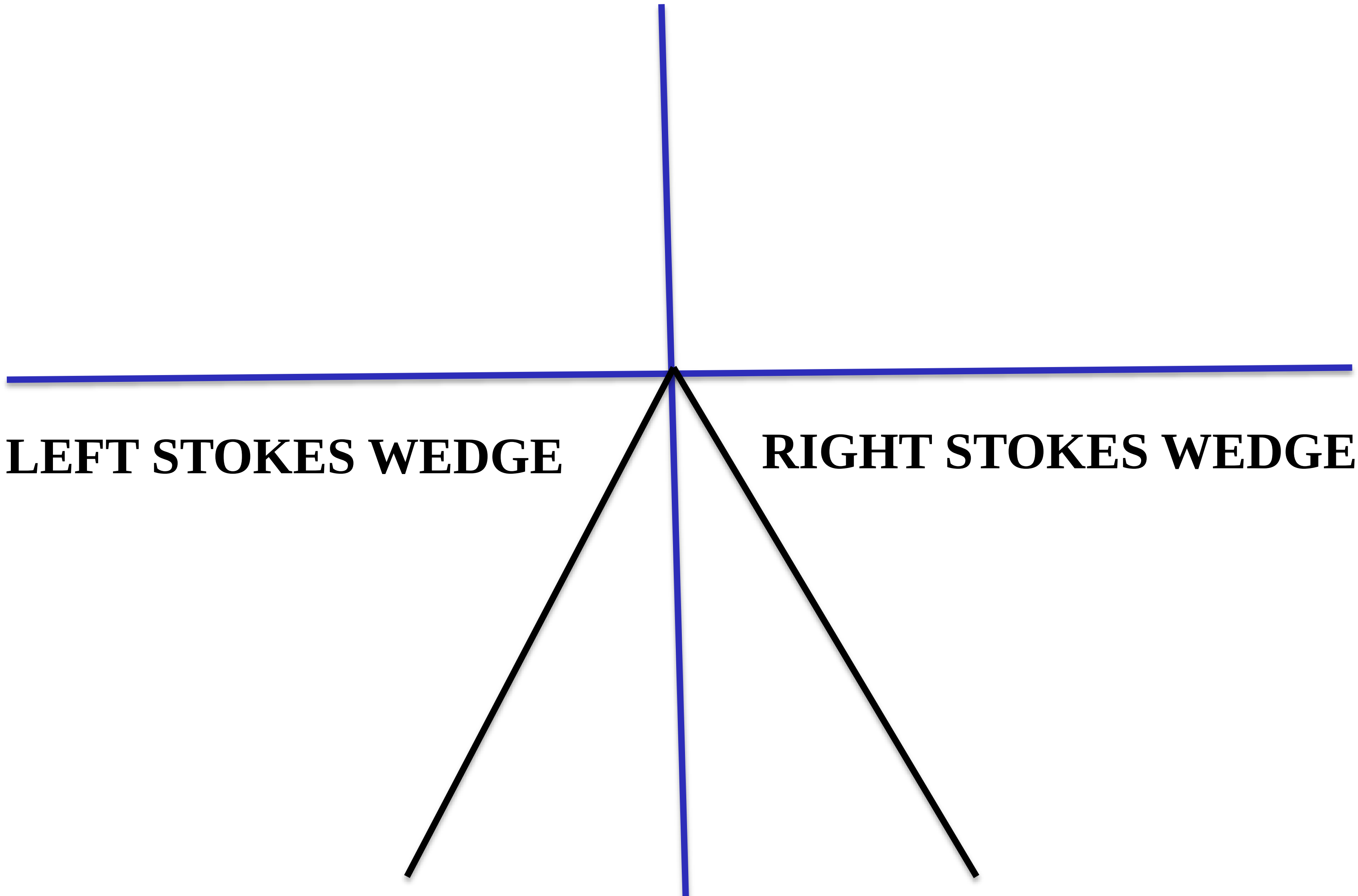
Because we have fallen into a trap!

We don’t want plane waves!
Plane waves give conventional quantum mechanics.

Reflectionless potentials

Unidirectional

Why is $-x^4$ reflectionless?



Eigenvalue problem

For the differential equation

$$-\phi''(x) - x^4\phi(x) = E\phi(x)$$

We get bound states at

$$E_0 = 1.477150$$
$$E_1 = 6.003386$$
$$E_2 = 11.802434$$

These bound states are determined by the vanishing of the solution in the left and right Stokes wedges.

WKB approximation

$$\phi_{WKB}(x) = C_\pm (E + x^4)^{-1/4} e^{\pm i \int^x ds \sqrt{s^4+E}}$$

$$\phi(x) \sim e^{\pm i x^3/3}$$

In *right* Stokes wedge

$$x = re^{i\theta} \qquad\qquad -\pi/3 < \theta < 0$$

So,

$$\phi(x) \sim e^{-i x^3/3}$$

In *left* Stokes wedge

$$-\pi < \theta < -2\pi/3$$

So, again

$$\phi(x) \sim e^{-ix^3/3}$$

Rotating to upper edges of the Stokes wedges on the real axis we have unidirectional waves going to the right

Boundary conditions

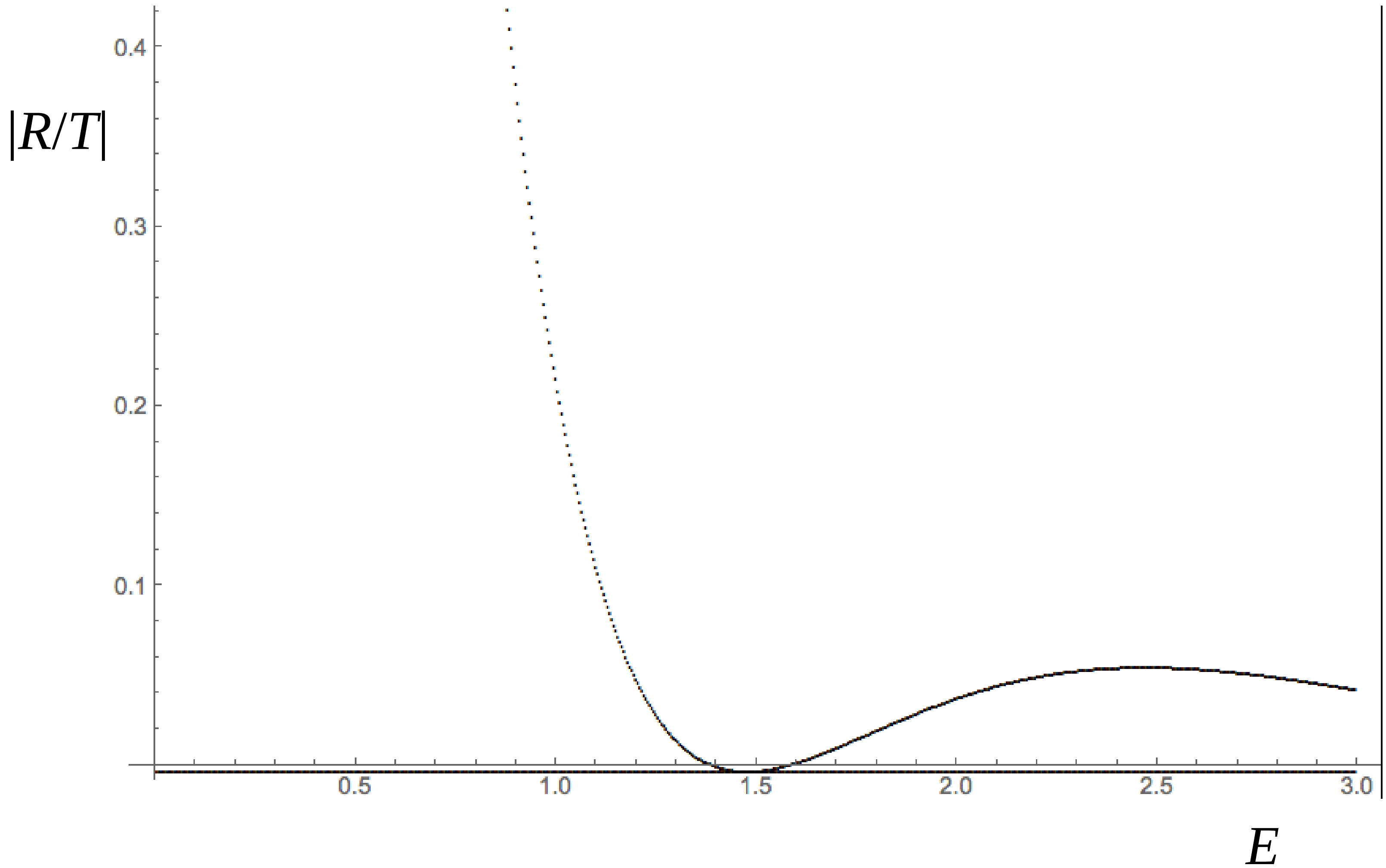
So, use the following boundary conditions: Choose the constant *C* so that on the positive-*x* axis at $x = L$

$$\phi_{III}(L) = 1$$

$$\phi'_{\mathrm{III}}(L) = -i\sqrt{L^4 + E}$$

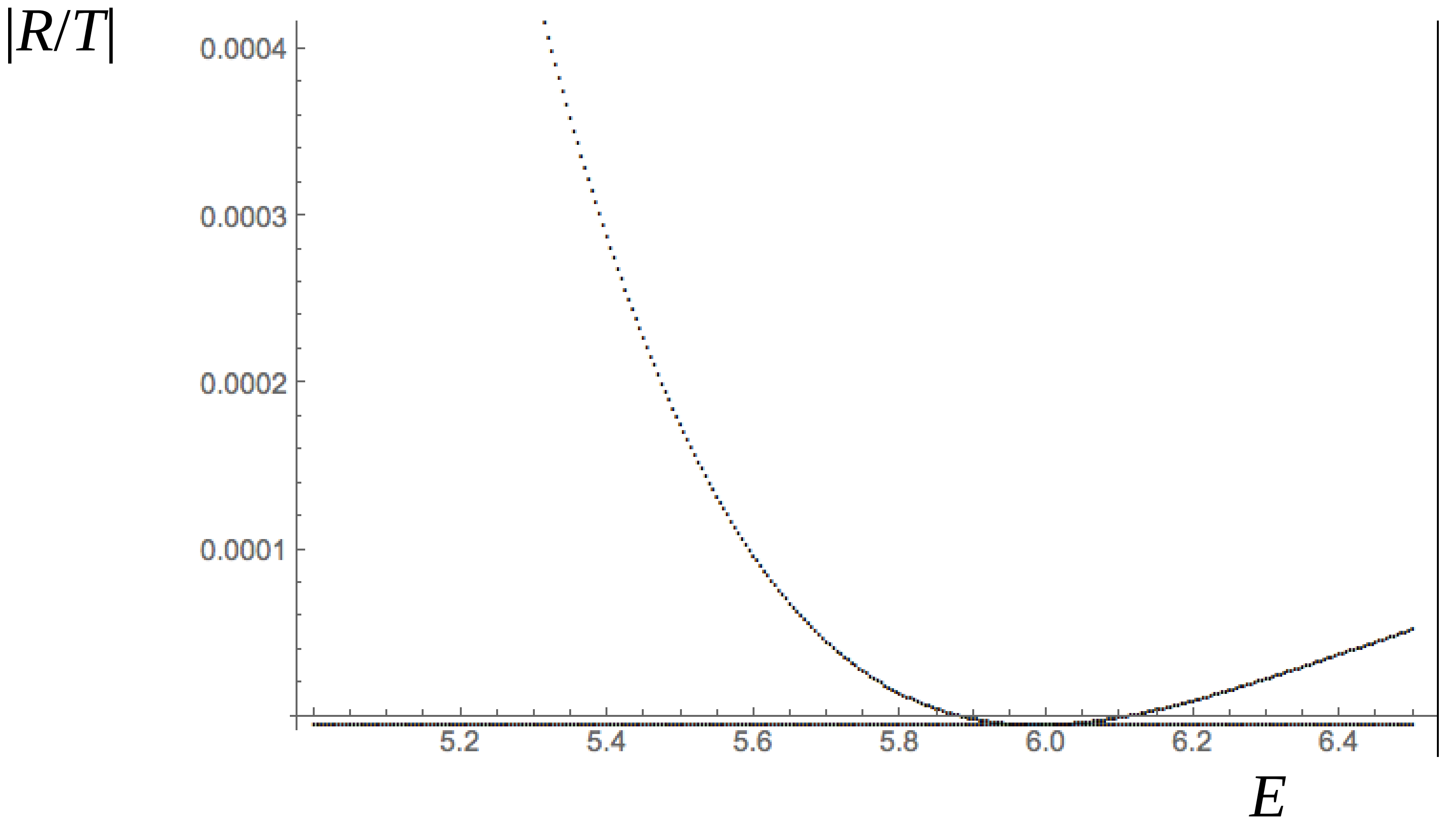
Then, on the negative-*x* axis at $x = -L$ we require that there be no reflected wave. Now we get the right energies!!

First eigenvalue



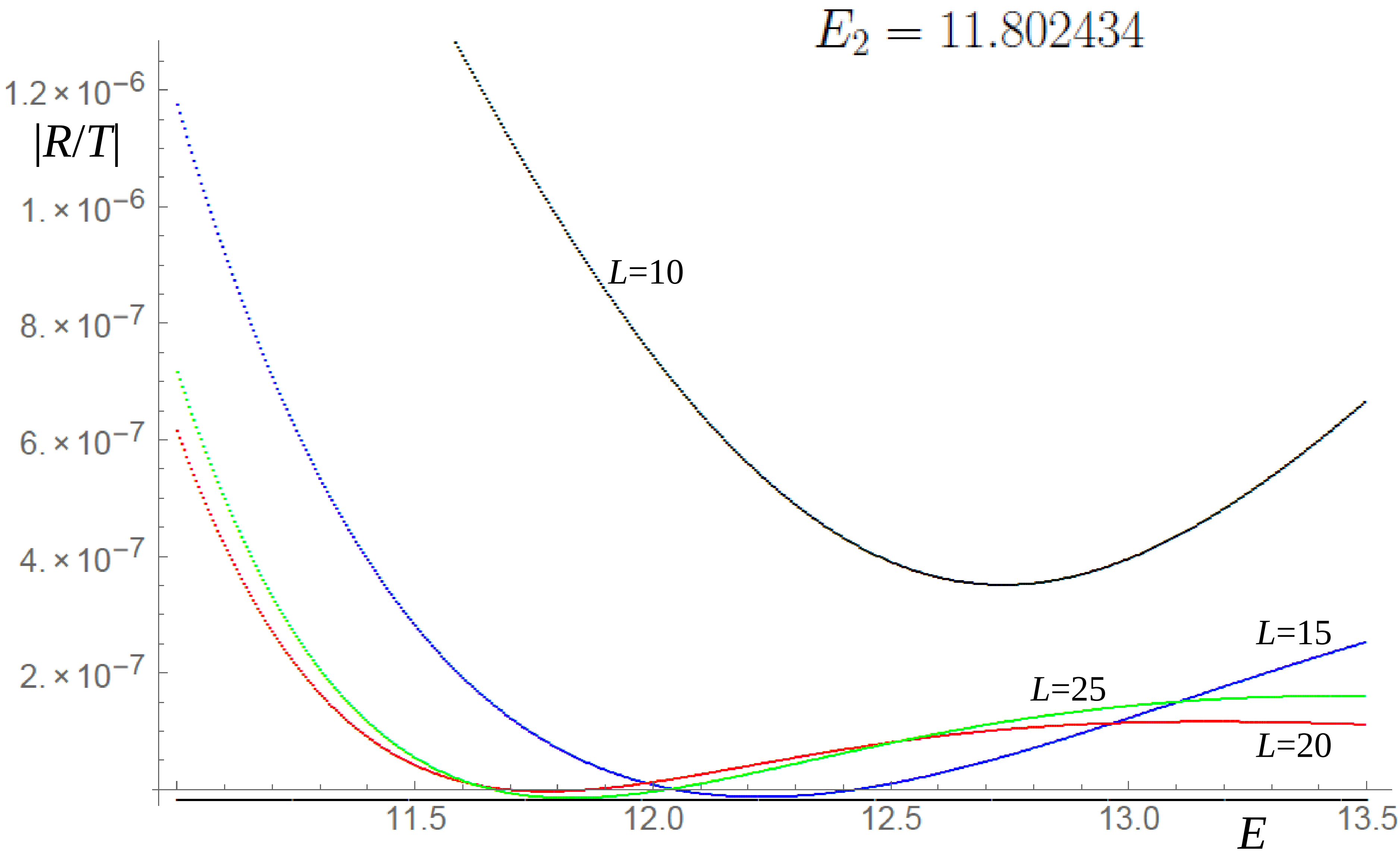
$$E_0 = 1.477150$$

Second eigenvalue



$$E_1 = 6.003386$$

Approach to the third eigenvalue



If we design the incident wave correctly we can see reflectionless behavior!

Thank you for listening!

Please work
on *PT* symmetry!