Information loss at exceptional points

Dorje C. Brody

Department of Mathematics
Brunel University London, Uxbridge UB8 3PH
dorje.brody@brunel.ac.uk

Non-Hermitian Photonics in Complex Media
Heraklion, Crete, 16-18 June 2016

The need for complex numbers:

1. Real analysis — insufficient (cf. Bender's talk)

2. Quantum mechanics

3. Thermal phase transitions (classical) — Lee-Yang theory

4. Quantum phase transitions

5. Systems with gain and loss (e.g., dissipation)

6. &c.

Thermal phase transition—van der Waals model in P-T distribution

The van der Waals equation of state:

$$\left(P + a\frac{N^2}{V^2}\right)(V - bN) = Nk_BT. \qquad (1)$$

The canonical partition function is

$$z(\beta, V) = \frac{1}{N!h^{3N}} \prod_{i=1}^{N} \int \mathrm{d}^3\mathrm{p}_i \int_V \mathrm{d}^3 r_i \exp\Big(-\beta \sum_{i=1}^{N} \frac{\mathrm{p}_i^2}{2m} - \beta \sum_{(ij)} \phi_{ij} \Big), \qquad (2)$$

where $\phi_{ij} = \phi(r_{ij})$, with the Lennard-Jones potential

$$\phi(r) = \phi_0 \left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6} \right]. \qquad (3)$$

The partition function in the pressure-temperature distribution is

$$Z(\alpha, \beta) = \frac{1}{b} \int_{bN}^{\infty} z(\beta, V) \exp(-\alpha V) \mathrm{d}V. \qquad (4)$$

The free energy $\psi(\alpha, \beta) = \lim_{n\to\infty} N^{-1} \ln Z(\alpha, \beta)$ can be calculated by use of the saddle-point method to yield (1).

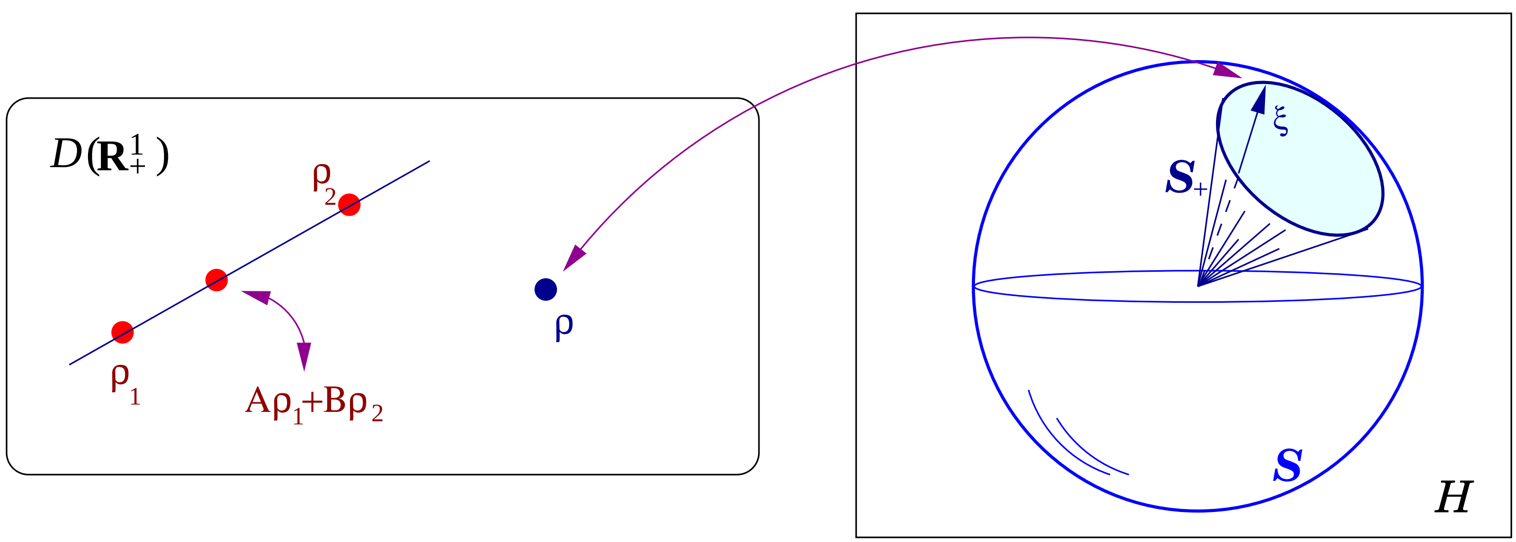
Hilbert-space formulation of statistical mechanics

Consider the embedding of the thermal density

$$\rho(x,V|\alpha,\beta) = \frac{\mathrm{e}^{-\beta H(x)-\alpha V}}{Z(\alpha,\beta)} \tag{5}$$

in a Hilbert space $\mathcal{H}$ of square-integrable functions according to:

$$\rho(x,V|\alpha,\beta) \to \xi(x,V|\alpha,\beta) = \sqrt{\rho(x,V|\alpha,\beta)} \equiv |\xi(\alpha,\beta)\rangle. \tag{6}$$



The parameter-space (α, β) inherits metric geometry from the ambient spherical geometry via:

$$G_{ij} = 4\langle \partial_i \xi(\alpha, \beta) | \partial_j \xi(\alpha, \beta) \rangle. \tag{7}$$

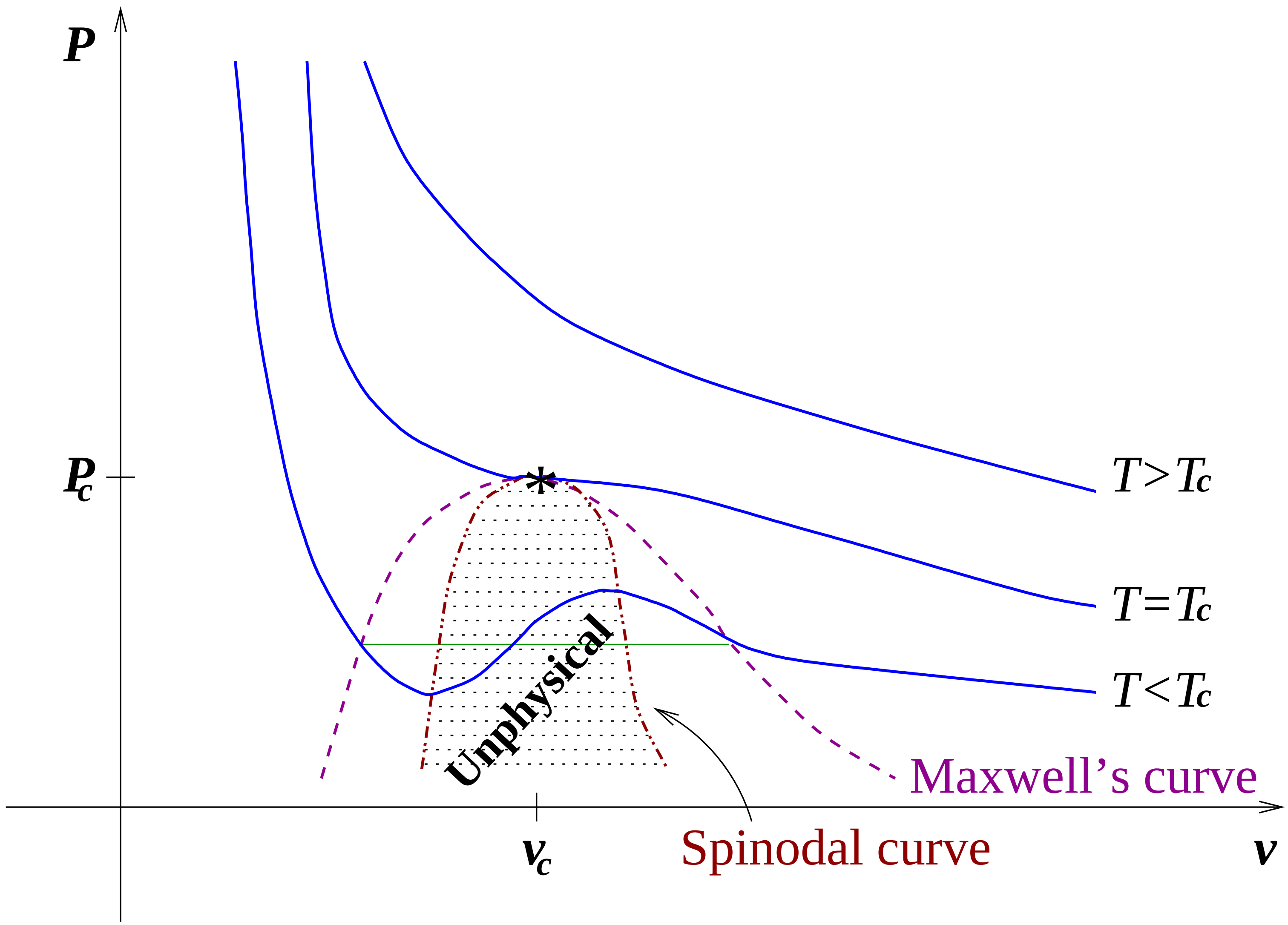
For the van der Waals gas we find:

1. Curvature singularity along the spinodal curve;

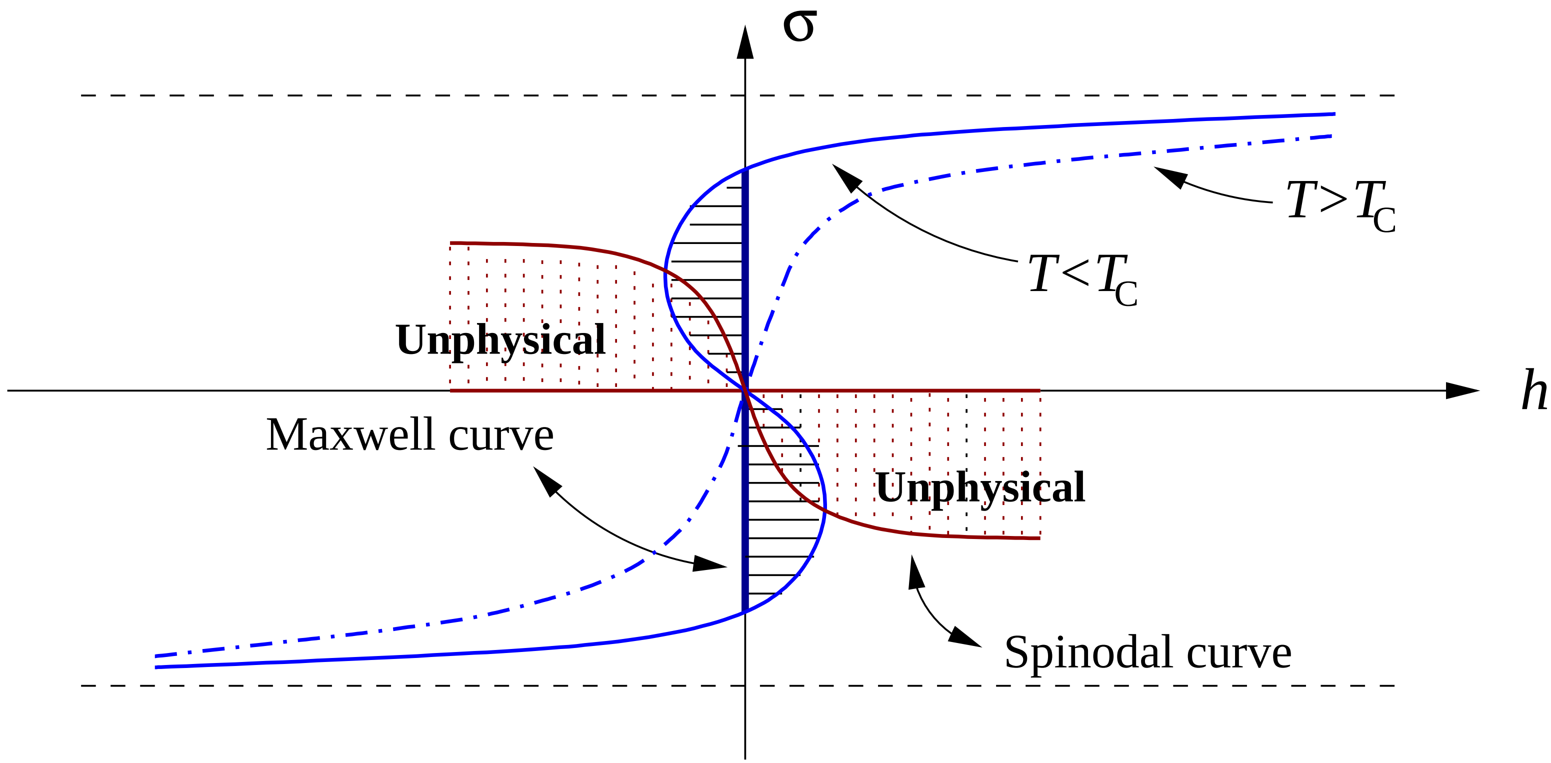
2. Information loss of the initial condition; and

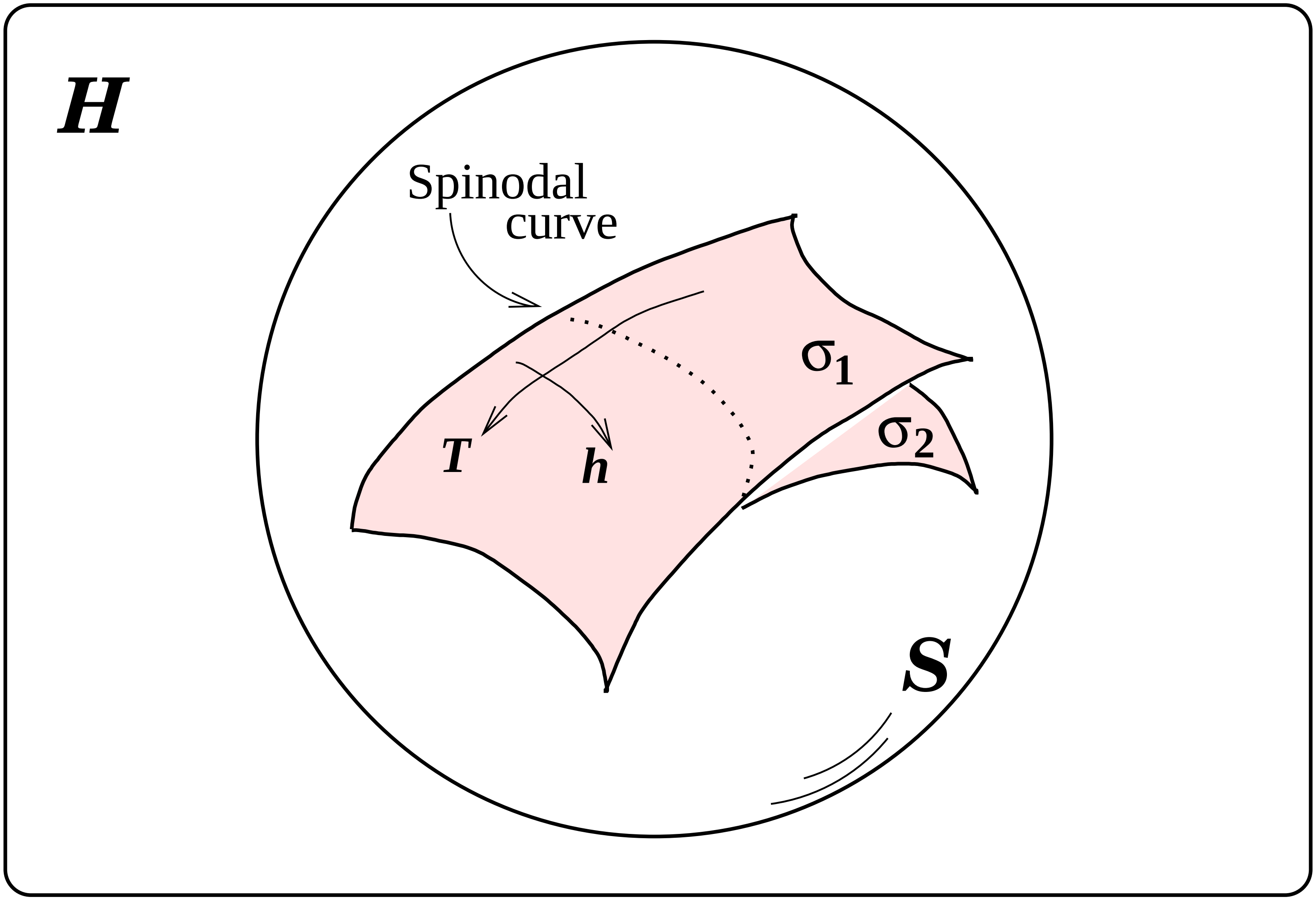
3. Breakdown of adiabaticity in the vicinity of the critical point

Some insights can be gained via the Lee-Yang theory...


P
P_c
*
$T>T_c$
$T=T_c$
$T<T_c$
Unphysical
Maxwell's curve
Spinodal curve
v_c
v

For the Ising magnet...




H
Spinodal
curve
T
h
σ_1
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S

Geometry in complex vector spaces

If a state $|\xi\rangle = |\xi(\theta)\rangle$ in $\mathbb{C}^n$ depends smoothly on a set of parameters $\{\theta^a\}$, then we have $|\mathrm{d}\xi\rangle = |\partial_a\xi\rangle\mathrm{d}\theta^a$, and we have

$$\mathrm{d}s^2 = 4\big(\langle\partial_a\xi|\partial_b\xi\rangle - \langle\xi|\partial_a\xi\rangle\langle\partial_b\xi|\xi\rangle\big)\mathrm{d}\theta^a\mathrm{d}\theta^b. \tag{8}$$

Equivalently, the metric on the parametric subspace is

$$G_{ab} = 4\left(\langle\partial_{(a}\xi|\partial_{b)}\xi\rangle - \langle\xi|\partial_{(a}\xi\rangle\langle\partial_{b)}\xi|\xi\rangle\right). \tag{9}$$

From G_{ab} one can calculate geometric quantities such as the Ricci curvature.

Eigenstates of Hermitian Hamiltonians

Consider a parametric family of Hermitian Hamiltonians $\hat{H}(\theta)$, $\theta = \{\theta^a\}$, with distinct eigenvalues:

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle. \tag{10}$$

Writing $\partial_a = \partial/\partial\theta^a$, first-order perturbation gives

$$|\partial_a\phi_n\rangle = \sum_{m\neq n} \frac{\langle\phi_m|\partial_a\hat{H}|\phi_n\rangle}{E_n - E_m}|\phi_m\rangle, \tag{11}$$

and hence we obtain

$$G_{ab}^{(n)} = 4\sum_{m\neq n} \frac{\langle\phi_n|\partial_{(a}\hat{H}|\phi_m\rangle\langle\phi_m|\partial_{b)}\hat{H}|\phi_n\rangle}{(E_n - E_m)^2} \tag{12}$$

for the metric (cf. Zanardi *et al*. PRL **99**, 100603, 2007; QPT in the XY model).

Meaning of the metric

To understand the meaning of the metric G_{ab}, consider the simple case where there is only one parameter θ.

Then we can generate a shift in the parameter using the unitary operator

$$\hat{U} = \sum_n |\phi_n(\theta + \mathrm{d}\theta)\rangle\langle\phi_n(\theta)|. \tag{13}$$

Evidently, $\hat{U}$ transports the state $|\phi_n(\theta)\rangle$ into $|\phi_n(\theta + \mathrm{d}\theta)\rangle$.

The generators of this evolution are then given by the observable

$$\hat{X} = \mathrm{i}(\partial_\theta \hat{U})\hat{U}^{-1}. \tag{14}$$

If $\lambda^{-1}\hat{X}$ is the self-adjoint operator generating the shift in θ so that $\mathrm{e}^{-\mathrm{i}\hat{X}\epsilon/\lambda}\phi(\theta) = \phi(\theta + \epsilon)$ for $\epsilon \ll 1$, then we find that $G = 4\Delta X^2/\lambda^2$ and that

$$\Delta\Theta\,\Delta X \geq \frac{\lambda}{2}. \tag{15}$$

Eigenstates of complex Hamiltonians and their adjoints

Let $\hat{K} = \hat{H} - \mathrm{i}\hat{\Gamma}$ be a complex Hamiltonian with normalised eigenstates $\{|\phi_n\rangle\}$ and nondegenerate eigenvalues $\{\kappa_n\}$:

$$\hat{K}|\phi_n\rangle = \kappa_n|\phi_n\rangle \quad \text{and} \quad \langle\phi_n|\hat{K}^\dagger = \bar{\kappa}_n\langle\phi_n|. \tag{16}$$

Additionally, it will be convenient to introduce normalised eigenstates of the adjoint matrix $\hat{K}^\dagger$:

$$\hat{K}^\dagger|\chi_n\rangle = \bar{\kappa}_n|\chi_n\rangle \quad \text{and} \quad \langle\chi_n|\hat{K} = \kappa_n\langle\chi_n|. \tag{17}$$

The reason for introducing the additional states $\{|\chi_n\rangle\}$ is because the eigenstates $\{|\phi_n\rangle\}$ of $\hat{K}$ are in general not orthogonal:

$$\langle\phi_m|\phi_n\rangle = 2\mathrm{i}\frac{\langle\phi_m|\hat{\Gamma}|\phi_n\rangle}{\bar{\kappa}_m - \kappa_n} = 2\frac{\langle\phi_m|\hat{H}|\phi_n\rangle}{\bar{\kappa}_m + \kappa_n} \tag{18}$$

for $m \neq n$

With the introduction of $\{|\chi_n\rangle\}$, we have:

$$\langle\chi_n|\phi_m\rangle = \delta_{nm}\langle\chi_n|\phi_n\rangle \quad \text{and} \quad \sum_n \frac{|\phi_n\rangle\langle\chi_n|}{\langle\chi_n|\phi_n\rangle} = \mathbb{1}. \tag{19}$$

If we define a 'projection' operator $\hat{\Pi}_n$ according to

$$\hat{\Pi}_n = \frac{|\phi_n\rangle\langle\chi_n|}{\langle\chi_n|\phi_n\rangle}, \tag{20}$$

then we have

$$\hat{K} = \sum_n \kappa_n \hat{\Pi}_n. \tag{21}$$

Geometry of complex Hamiltonians

We now consider the perturbation of the eigenstates away from degeneracies:

$$(\hat{K}+\partial_a\hat{K}\mathrm{d}\theta^a+\cdots)(|\phi_n\rangle+|\partial_a\phi_n\rangle\mathrm{d}\theta^a+\cdots)=(\kappa_n+\partial_a\kappa_n\mathrm{d}\theta^a+\cdots)(|\phi_n\rangle+|\partial_a\phi_n\rangle\mathrm{d}\theta^a\cdots$$

Equating the terms linear in $\mathrm{d}\theta$ we find

$$(\hat{K}-\kappa_n)|\partial_a\phi_n\rangle=\partial_a\kappa_n|\phi_n\rangle-\partial_a\hat{K}|\phi_n\rangle. \tag{23}$$

If we multiply $\hat{\Pi}_m$ from the left and rearrange terms we find

$$(\kappa_m-\kappa_n)\hat{\Pi}_m|\partial_a\phi_n\rangle=(\partial_a\kappa_n)\delta_{mn}|\phi_m\rangle-\frac{\langle\chi_m|\partial_a\hat{K}|\phi_n\rangle}{\langle\chi_m|\phi_m\rangle}|\phi_m\rangle. \tag{24}$$

For $n=m$ we are led to the expression:

$$\partial_a\kappa_n=\frac{\langle\chi_n|\partial_a\hat{K}|\phi_n\rangle}{\langle\chi_n|\phi_n\rangle}. \tag{25}$$

For $n\neq m$ we divide both sides of (24) by $\kappa_m-\kappa_n$ and sum over $m\neq n$ to obtain

$$|\partial_a\phi_n\rangle=\sum_{m\neq n}\frac{\langle\chi_m|\partial_a\hat{K}|\phi_n\rangle}{\kappa_n-\kappa_m}|\phi_m\rangle. \tag{26}$$

The metric geometry of the parameter space can now be determined:

$$G_{ab} = 4\sum_{m\neq n} \frac{\langle\chi_m|\partial_{(a}\hat{K}|\phi_n\rangle\langle\phi_n|\partial_{b)}\hat{K}|\chi_m\rangle}{(\bar{\kappa}_n-\bar{\kappa}_m)(\kappa_n-\kappa_m)}. \tag{27}$$

The approach here breaks down as one comes close to exceptional points.

To obtain a 'higher precision' analysis in the vicinities of exceptional points we are required to go beyond the Rayleigh-Schrödingier perturbation theory, e.g., using the Newton-Puiseux series.

An explicit example

As an example, take the 2×2 PT-symmetric Hamiltonian $\hat{K} = \hat{\sigma}_x - \mathrm{i}\gamma\hat{\sigma}_z$.

This Hamiltonian has real eigenvalues in the region $\gamma^2 < 1$.

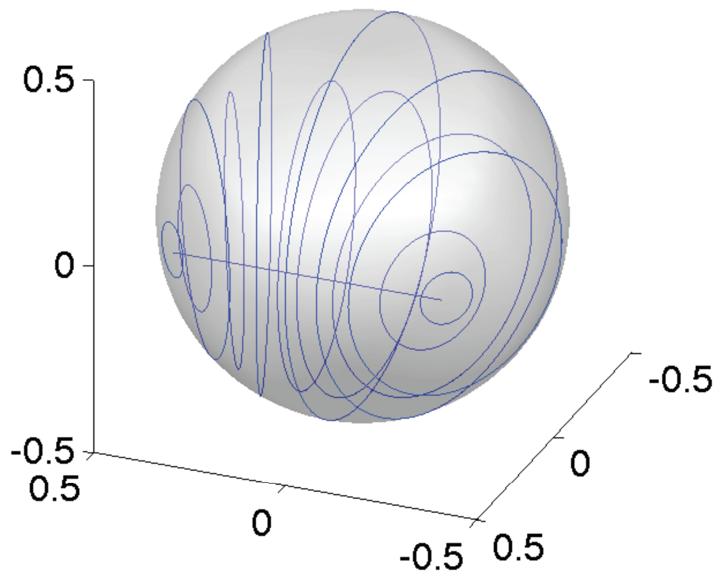
Specifically, the eigenstates of $\hat{K}$ and $\hat{K}^\dagger$ are given by

$$|\phi_\pm\rangle = n_\pm \begin{pmatrix} 1 \\ \mathrm{i}\gamma \pm \sqrt{1-\gamma^2} \end{pmatrix}, \qquad |\chi_\pm\rangle = n_\mp \begin{pmatrix} 1 \\ -\mathrm{i}\gamma \pm \sqrt{1-\gamma^2} \end{pmatrix}, \quad (28)$$

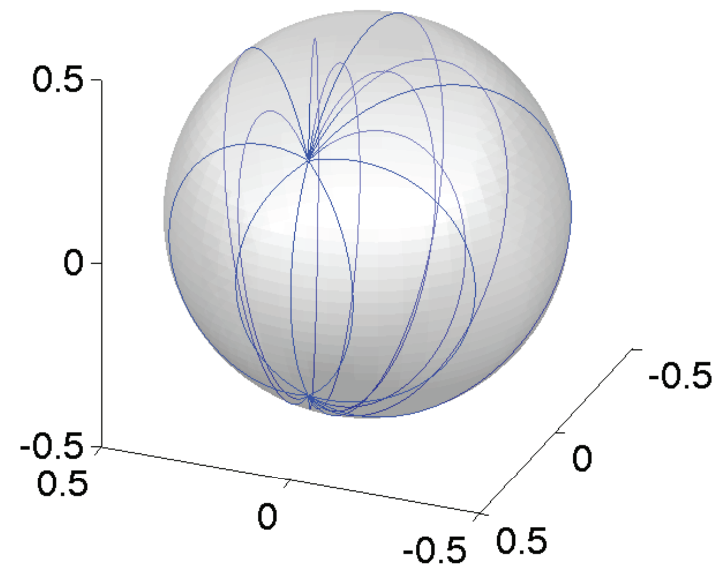
where $n_\pm^2 = (1 \mp \mathrm{i}\gamma/\sqrt{1-\gamma^2})/2$.

The "dynamics" of the state exhibit different characteristics for $\gamma^2 \leq 1$ and $\gamma^2 > 1$.

In particular, the time average of the order parameter $\mathrm{tr}(\hat{\sigma}_z\hat{\rho}_t)$ shows the existence of a phase transition at the critical point $\gamma_c = 1$



(a) unbroken phase



(b) broken phase

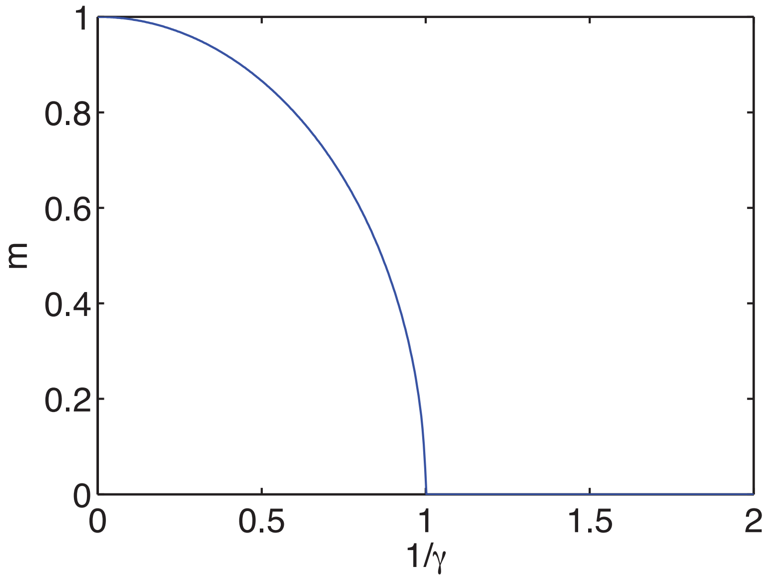
The information metric associated with the curve, say, $|\phi_+(\gamma)\rangle$, is given by

$$G = \frac{1}{(1-\gamma^2)^2}, \tag{29}$$

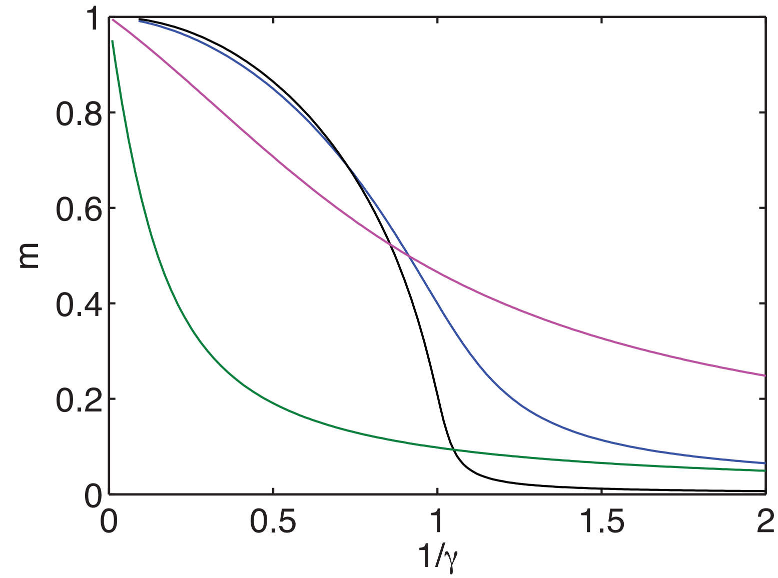
on account of the relations:

$$|\mathrm{d}\phi_+\rangle = -\frac{\mathrm{i}\,\mathrm{d}\gamma}{2(1-\gamma^2)}|\phi_-\rangle, \qquad \langle\widetilde{\mathrm{d}\phi_+}| = \frac{\mathrm{i}\,\mathrm{d}\gamma}{2(1-\gamma^2)}\langle\chi_-|. \tag{30}$$

(29) shows how the metric diverges as one approaches the critical point $\gamma_c = 1$.



(c) $\kappa = 0$



(d) $\kappa \neq 0$

More generally, any curve of the form

$$|\psi(\gamma)\rangle = c_+|\phi_+(\gamma)\rangle + c_-|\phi_-(\gamma)\rangle \tag{31}$$

with fixed coefficients $c_\pm$ in this system possesses the metric (29) and will exhibit a curvature singularity at $\gamma = 1$.

In the region $\gamma^2 \gg 1$, on the other hand, we have $G \ll 1$, and thus estimation of the parameter γ becomes unfeasible.

Discussion

There are features that are common to (i) thermal phase transitions; (ii) quantum phase transitions; and (iii) gain/loss (PT) transitions:

1. Loss of information (pure state $\Longrightarrow$ mixed state) is a generic feature.
2. Curvature singularity is a generic feature.
3. The breakdown of the adiabatic theorem is a generic feature.
4. The need for complex numbers to gain better understanding is a generic feature.

These features are reminiscent of the information-paradox of black holes...

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